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Jan. 1785*

INTRODUCTION

TO

A L G E B R A;

WITH

NOTES AND OBSERVATIONS:

DESIGNED FOR THE USE OF

SCHOOLS, and Places of PUBLIC EDUCATION.

By JOHN BONNYCASTLE, *K*

Author of the SCHOLAR'S GUIDE TO ARITHMETIC, and a
TREATISE OF MENSURATION.

—Ingenuas didicisse fideliter artes
Emollit mores, nec finit esse feros.

OVID.

L O N D O N:

Printed for J. JOHNSON, No. 72, ST. PAUL'S-CHURCH-
YARD, 1782.

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Printed for J. JOHNSON, No. 25, St. Paul's Church-
Yard, 1782.

TO THE REV.

J. PRIESTLEY, L.L.D. F.R.S. &c.

SIR,

THE publication of the following treatise is owing to your kind encouragement and approbation; and I am happy to embrace this opportunity of testifying the high sense I entertain of your condescending politeness and attention. Whilst you are enlarging the bounds of science by your numerous and important discoveries, you are equally solicitous of promoting every other laudable pursuit and useful undertaking. And to this amiable disposition the world is no less indebted than to

your distinguished and eminent abilities : the one commands our esteem and regard, and the other our admiration. Permit me, therefore, Sir, as a sincere tribute to your merit, to inscribe to you this compendium, and to assure you that

I am,

With the highest respect,

Your most obedient and

obliged humble servant,

London,
Sep. 2, 1782.

JOHN BONNYCASTLE.

P R E F A C E.

THE powers of the mind, like those of the body, are increased by continual exertion; application and industry supply the place of genius and invention; and even the creative faculty itself, may be strengthened and improved by use and perseverance. Uncultivated nature is uniformly rude and imbecile; and it is by imitation alone that we at first acquire knowledge, and the means of extending its bounds. A just and perfect acquaintance with the simple elements of science, is a necessary step towards our future progress and advancement; and this, assisted by laborious investigation and habitual inquiry, will constantly lead to eminence and perfection.

Books of rudiments, therefore, concisely written, well digested, and methodically arranged, are treasures of inestimable value; and too many attempts cannot be made to render them perfect and complete. When the first principles of any art or science are firmly fixed and rooted in the mind, their application soon becomes easy,

pleasant, and obvious; the understanding is delighted and enlarged; we conceive clearly, reason distinctly, and form just and satisfactory conclusions. But, on the contrary, when the mind, instead of reposing on the stability of truth, and received principles, is wandering in doubt and uncertainty, our ideas will necessarily be confused and obscure; and every step we take, must be attended with fresh difficulties and endless perplexity.

That the grounds, or fundamental parts, of every science, are dull and unentertaining, is a complaint universally made, and a truth not to be denied; but, then, what is obtained with difficulty is remembered with ease; and what is purchased with pain is possessed with pleasure. The seeds of knowledge are sown in every soil, but it is by proper culture alone that they are cherished and brought to maturity. A few years of early and assiduous application never fails of procuring us the reward of our industry; and who is there, that knows the pleasures and advantages which the sciences afford, that would think his time mis-spent, or his labours useless? Riches and honours are the gifts of fortune, casually bestowed or hereditarily received; and are frequently

abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, that originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But as the modes, situations, and circumstances of life are various, so accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the listless and dormant powers of younger minds, we have only to point out to them a valuable acquisition, and the means of obtaining it. The active principles are immediately put into motion, and the certainty of the conquest is insured from a determination to conquer.

But of all the sciences which serve to call forth this spirit of enterprise and inquiry, there are none more eminently useful than the Mathematics.

By an early attachment to these elegant and sublime studies, we acquire a habit of reasoning, and an elevation of thought, that fixes the mind, and prepares it for every other pursuit. From a few simple axioms, and evident principles, we proceed gradually to the most general propositions, and remote analogies; deducing one truth from another, in a chain of argument, well connected and logically pursued; that brings us at last, in the most satisfactory manner, to the conclusion, and serves as a general direction in all our inquiries after truth.

And it is not only in this respect that mathematical learning is so highly valuable: it is, likewise, equally estimable for its practical utility. Almost all the works of art, and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of these admirable sciences is, therefore, a thing of the utmost importance, and ought to be considered as a principal part of every liberal and well regulated plan of education. They are the guide of our youth, the perfection of our reason, and the foundation, or basis, of every great and noble undertaking.

From these considerations, I have been induced to undertake an introductory course of mathematical science; and, from the kind encouragement I have hitherto received, am not without hopes of a continuance of the same candour and approbation. Considerable practice as a teacher, and a long attention to the difficulties and obstructions which retard the progress of learners in general, has enabled me to accommodate myself the more easily to their capacities and understandings. And as an earnest desire of promoting and diffusing useful knowledge, is the chief motive for this undertaking, so no pains, or attention, shall be wanting to make it as complete and perfect as possible.

The subject of the present performance is ALGEBRA; which is one of the most important and useful branches of those sciences; and may be justly considered as the key to all the rest. Geometry delights us by the simplicity of its principles, and the elegance of its demonstrations: Arithmetic is confined in its object, and partial in its application: but Algebra, or the Analytic Art, is general and comprehensive, and may be used with success, in all cases, where truth is to be obtained, and proper data can be established.

To trace this noble science to its birth, and to point out all the various alterations and improvements it has undergone in its progress, would far exceed the limits of a preface. It will be sufficient to observe, that the invention is of the highest antiquity, and has challenged the praise and admiration of all ages. *Diophantus* appears to have been the first, among the ancients, who applied it to the solution of indeterminate and unlimited problems; but it is to the moderns that we are principally indebted for all the most curious refinements of the art, and its great and extensive usefulness in every abstruse and difficult inquiry. *Newton*, *Maclaurin*, *Saunderson*, *Simpson*, and *Emerson*, are those, of our own countrymen, who have particularly excelled in this respect; and it is to their works that I would refer the young student, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of those writers, and is intended as a useful and necessary introduction to them. Almost every subject, that belongs to pure Algebra, is concisely and distinctly treated of; and no pains have been spared, to make the whole as easy and intelligible as possible. A great number of elementary books have already been written

P R E F A C E.

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upon this subject; but there are none, that I have yet seen, but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste, and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is destroyed by the clumsy and awkward manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged to unlearn and forget every thing that he has been at so much pains in acquiring.

It is in the sciences as in every branch of polite literature; there is a certain taste and elegance that is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the business of every writer who engages in the humble, but useful, task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it could be thought to be in the least useful or necessary. The nature and confined limits of my plan,

admitted not of diffuse observations, or a formal enumeration of particulars; but nothing of real use and importance has been omitted. My principal object was to consult the ease, satisfaction, and accommodation of the learner, and if the execution of the work is found equal to the design, my purpose will be answered, and it cannot fail of meeting with a candid and favourable reception from the public.

5 DEGO

E R R A T A.

Page 12, line 7, for xy^4 read $x^4 - y^4$. P. 20, l. 13, for $2bx^2$ read $2bx$. P. 22, l. 3, for $a + b$ read $x + b$. P. 27, l. 13, for $\frac{5bx}{a}$ read $\frac{5bx}{2a}$. P. 29, l. 29, dele —. P. 31, l. 20, for $x + b$ read $x + a$. P. 32, l. 10, dele —. P. 49, l. 5, for x^5 read x^6 . P. 50, l. 24 and 26, read $a \times^3 \sqrt{a}$, $3a^3 \times^3 \sqrt{a}$, $9a^5 \times^3 \sqrt{a}$, &c. P. 54, l. 33, for $+$ read $::$. P. 56, l. 11, for 19 read 16. P. 60, l. 24, for $+ 3x$ read $- 3x$, and l. ult. for ba read $6a$. P. 64, l. 9, for 2 read 3. P. 79, l. 26, dele $\sqrt{}$. P. 82, l. 21, for $x -$ read $x =$, and line ult. for negative read affirmative.

A L G E B R A.

DEFINITIONS.

ALGEBRA is the art of computing by symbols.

1. *Like quantities* are those that consist of the same letters.
2. *Unlike quantities* are those that consist of different letters.
3. *Given quantities* are those whose values are known.
4. *Unknown quantities* are those whose values are unknown.
5. *Simple quantities* are those that consist of one term only.
6. *Compound quantities* are those that consist of several terms.
7. *Positive or affirmative quantities* are those to be added.
8. *Negative quantities* are those to be subtracted.
9. *Like signs* are all $+$ or all $-$.
10. *Unlike signs* are $+$ and $-$.

2 EXPLANATION OF

11. *The co-efficient* of any quantity is the number prefixed to it.

12. *A binomial quantity* is one consisting of two terms; a *trinomial* of three terms; and a *quadrinomial* of four terms; &c.

13. *A residual quantity* is a binomial where one of the terms is negative.

14. *The power of a quantity* is its square, cube, biquadrate &c.

15. *The index or exponent* is the number expressing the power to which the quantity is involved.

16. *A rational quantity* is that which has no radical sign.

17. *The reciprocal* of any quantity is that quantity inverted, or unity divided thereby.

EXPLANATION OF THE CHARACTERS.

+	Is the sign of addition.
-	— of subtraction.
×	— of multiplication.
÷	— of division.
√	— of the square root.
³ √	— of the cube root.
^m √	— of the <i>m</i> root.
=	— of equality.

Thus $a+b$ is the sum of a and b .

$a-b$ is the difference of a and b .

$a \propto b$ is the difference of a and b when it is not known which is the greatest.

ab , or $a \times b$, or $a.b$ is the product of a and b .

$a \div b$ or $\frac{a}{b}$ is a divided by b .

$\sqrt{a}$ or $a^{\frac{1}{2}}$ is the square root of a .

$\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ is the cube root of a .

a^2 is the square of a .

a^3 is the cube of a .

a^m is the m power of a .

$a^{\frac{1}{m}}$ is the m root of a .

$\frac{1}{a}$ is the reciprocal of a , and $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$.

In the computation of problems, the first letters of the alphabet are put for known quantities, and the last letters for those that are unknown.

A X I O M S.

1. If equal quantities be added to equal quantities the wholes will be equal.

2. If equal quantities be taken from equal quantities, the remainders will be equal.

3. If equal quantities be multiplied by equal quantities, the products will be equal.

4. If equal quantities be divided by equal quantities, the quotients will be equal.

5. The equal powers or roots of equal quantities are equal.

6. Two quantities respectively equal to a third, are equal to each other.

7. The whole is equal to all its parts taken together.

A D D I T I O N.

C A S E I.

To add quantities that are like, and have like signs.

R U L E.

Add all the co-efficients together, and to their sum adjoin the letters common to each term, prefixing the common sign.

E X A M P L E S * :

5 a	— 6 bx	8 bxy
7 a	— 3 bx	7 bxy
8 a	— 2 bx	3 bxy
10 a	— 7 bx	4 bxy
2 a	— bx	5 bxy
a	— 5 bx	bxy
33 a	— 24 bx	28 bxy

5 x ² + xy	7 ax — y
3 x ² + 2 xy	8 ax — 3 y
x ² + 3 xy	6 ax — 2 y
7 x ² + 8 xy	4 ax — 3 y
x ² + xy	ax — y
17 x ² + 15 xy	26 ax — 10 y

* When a leading quantity has no sign before it + is always understood; and a quantity without any co-efficient prefixed to it is supposed to have 1, or unity.

$$\begin{array}{r}
 15xy \\
 2xy \\
 7xy \\
 xy \\
 xy \\
 \hline
 \end{array}
 \begin{array}{r}
 -8y^2 \\
 -7y^2 \\
 -y^2 \\
 -6y^2 \\
 -y^2 \\
 \hline
 \end{array}
 \begin{array}{r}
 7a-6b \\
 4a-3b \\
 2a-8b \\
 a-b \\
 3a-2b \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 3x^{\frac{1}{2}}-xy \\
 2x^{\frac{1}{2}}-3xy \\
 4x^{\frac{1}{2}}-8xy \\
 x^{\frac{1}{2}}-2xy \\
 \hline
 \end{array}
 \begin{array}{r}
 3xy-x+2ab \\
 2xy-3x+2ab \\
 2xy-4x+8ab \\
 5xy-3x+ab \\
 \hline
 \end{array}$$

CASE II.

To add quantities that are like, but have unlike signs.

R U L E.

1. Add all the affirmative co-efficients into one sum, and all the negative ones into another.
2. Subtract the least sum from the greatest, and to the difference prefix the sign of the greatest, with the common quantity.

EXAMPLES:

$-3a$	$+8ax^2$	$+6x^{\frac{1}{3}}+8y$
$+7a$	$+7ax^2$	$-3x^{\frac{1}{3}}+7y$
$+8a$	$-3ax^2$	$-13x^{\frac{1}{3}}+8y$
$-a$	$-4ax^2$	$+2x^{\frac{1}{3}}-3y$
$-2a$	$+4ax^2$	$+x^{\frac{1}{3}}-y$
$+9a$	$+12ax^2$	$-7x^{\frac{1}{3}}+19y$

$-2xy+8$	$+8x^2-y+3\sqrt{x}$
$-3xy+7$	$-10x^2-3y+2\sqrt{x}$
$+xy-10$	$-4x^2-2y+\sqrt{x}$
$+5xy-7$	$+9x^2+6y-10\sqrt{x}$
$+xy-2$	$+3x^2 \dots -4\sqrt{x}$

$-2a^2$	$+8b^2y^3$	$-3ab+3$
$-3a^2$	$+6b^2y^3$	$+8ab-10$
$-8a^2$	$-10b^2y^3$	$+3ab-6$
$+10a^2$	$-20b^2y^3$	$-ab+2$
$+13a^2$	$-b^2y^3$	$-2ab+11$

$+4\sqrt{bx}-x$	$+4x^2-y^2-2$
$-3\sqrt{bx}-3x$	$+10x^2-8y^2-4$
$-\sqrt{bx}-8x$	$-6x^2-y^2+8$
$-6\sqrt{bx}+11x$	$+3x^2+10y^2-1$

CASE III.

To add quantities that are unlike, and have unlike signs.

RULE.

Collect the like quantities together by the last rule, and set down those that are unlike, one after another, with their proper signs.

EXAMPLES;

$$\begin{array}{rcl}
 2x & & 2x - x^2 \\
 3y & & 3a - x \\
 -a & & 2ax + 6x^2 \\
 x^2 & & 3\sqrt{x} - 2ax \\
 \hline
 2x + 3y - a + x^2 & & x + 5x^2 + 3a + 3\sqrt{x} \\
 \hline
 \end{array}$$

$$\begin{array}{rcl}
 12ax - x^2 - 6 + \sqrt{ax} - x^2 & & \\
 -6ax + x^2 - x + 10 & & \\
 3y - ax - 4 - 2\sqrt{ax} - x^2 & & \\
 \hline
 5ax - \sqrt{ax} - x^2 - x + 3y & & \\
 \hline
 \end{array}$$

$$\begin{array}{rcl}
 3x^2y & 2\sqrt{x} - 8 & 3a^2 - 8 + x^{\frac{1}{2}} - 2a \\
 -2xy & 3\sqrt{xy} + 10 & 3a - 10 + a^2 - 6a \\
 -3y^2x & 6ab + x & x^2 - 4a^2 + 8 - 4\sqrt{x} \\
 -8x^2y & -8 + \sqrt{xy} & 10 - a - x^2 - x^{\frac{1}{2}} \\
 \hline
 \hline
 \hline
 \end{array}$$

SUBTRACTION.

RULE.

Change the signs of all the quantities to be subtracted, and then add them together as in addition, and the result will be the remainder required.

EXAMPLES:

$$\begin{array}{r} 3a^2 - 2b \\ 2a^2 - 3b \\ \hline a^2 + b \end{array} \quad \begin{array}{r} 6x^2 - 8y + 2 \\ x^2 + 9y - 20 \\ \hline 5x^2 - 17y + 22 \end{array} \quad \begin{array}{r} 35xy - 2 + 8x - y^{\frac{1}{2}} \\ 24xy - 8 - 8x - 3y \\ \hline 11xy + 6 + 16x - y^{\frac{1}{2}} + 3y \end{array}$$

$$\begin{array}{r} 8ax - 2\sqrt{xy} - 10 \\ 10x - 6\sqrt{xy} - ax \\ \hline 9ax + 4\sqrt{xy} - 10 - 10x \end{array} \quad \begin{array}{r} 4\sqrt{\frac{x}{a}} - 10 - 8x - 3xy \\ -5xy - 7x + 3 - y \\ \hline 4\sqrt{\frac{x}{a}} - 13 - x + 2xy + y \end{array}$$

$$\begin{array}{r} 5x^2y - 8 \\ -3x^2y + 10 \\ \hline \end{array} \quad \begin{array}{r} 4\sqrt{xy} - x\sqrt{xy} \\ 2\sqrt{xy} + 12 + xy \\ \hline \end{array} \quad \begin{array}{r} 5x^2y^2 + \sqrt{\frac{x}{a}} - 8 - 4b \\ 6x^2y^2 - 10 + 4b - \sqrt{\frac{x}{2}} \\ \hline \end{array}$$

MULTIPLICATION.

CASE I.

To multiply simple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product prefix all the letters in those terms, and the result will be the whole product required.

Note. Like signs produce +, and unlike signs —.

EXAMPLES:

$2a$	$-2a$	$5a$	$-9x$
$3b$	$+4b$	$-6x$	$-5b$
$6ab$	$-8ab$	$-30ax$	$+45bx$

$7ab$	$6a^2x$	$-x^2y$	$-7xy$
$-5ac$	$5x$	xy^2	$-xy$
$-35a^2bc$	$30a^2x^2$	$-x^3y^3$	$+7x^2y^2$

a	$-ax$	$+5xy$	$-xy$
x	$-7b$	-3	$+6ax$

CASE II.

When one of the factors is a compound quantity.

R U L E.

Find the products of the multiplier, and every particular term of the multiplicand separately, and place them one after another, with their proper signs, and the result will be the whole product required.

E X A M P L E S :

$4a-2b$	$6xy-8$	$8a^2-2x+6$
$3a$	$2x$	$3xy$
$12a^2-6ab$	$12x^2y-16x$	$24a^2xy-6x^2y+18xy$

$3x-a$	$5x-7a$	$3y-8+2xy$
$2a$	x	xy

CASE III.

When both the factors are compound quantities.

R U L E.

Multiply every particular term of the multiplier into every term of the multiplicand respectively, and set down the products one after another with their proper signs, and their sum will be the whole product required.

MULTIPLICATION. 11

EXAMPLES:

$$\begin{array}{r} x+y \\ x+y \\ \hline \end{array}$$

$$\begin{array}{r} x^2+xy \\ +xy+y^2 \\ \hline \end{array}$$

$$\hline x^2+2xy+y^2$$

$$\begin{array}{r} x+y \\ x-y \\ \hline \end{array}$$

$$\begin{array}{r} x^2+xy \\ -xy-y^2 \\ \hline \end{array}$$

$$\hline x^2 \quad * \quad -y^2$$

$$\begin{array}{r} x-y \\ x-y \\ \hline \end{array}$$

$$\begin{array}{r} x^2-xy \\ -xy+y^2 \\ \hline \end{array}$$

$$\hline x^2-2xy+y^2$$

$$\begin{array}{r} 5x+4y \\ 3x-2y \\ \hline \end{array}$$

$$\begin{array}{r} 15x^2+12xy \\ -10xy-8y^2 \\ \hline \end{array}$$

$$\hline 15x^2+2xy-8y^2$$

$$\begin{array}{r} x^2+xy-y^2 \\ x-y \\ \hline \end{array}$$

$$\begin{array}{r} x^3+x^2y-xy^2 \\ -x^2y-xy^2-y^3 \\ \hline \end{array}$$

$$\hline x^3 \quad * \quad 2xy^2-y^3$$

$$\begin{array}{r} x^2+xy+y^2 \\ x-y \\ \hline \end{array}$$

$$\begin{array}{r} x^3+x^2y+xy^2 \\ -x^2y-xy^2-y^3 \\ \hline \end{array}$$

$$\hline x^3 \quad * \quad * \quad -y^3$$

$$\begin{array}{r} x+y-z \\ x-y+z \\ \hline \end{array}$$

$$\begin{array}{r} x^2+xy-xz \\ -xy-y^2+yz \\ +xz+yz-z^2 \\ \hline \end{array}$$

$$\hline x^2 \quad * \quad -y^2+2yz-z^2$$

$$\begin{array}{r} 3x^2-2xy+5 \\ x^2+2xy-3 \\ \hline \end{array}$$

$$\begin{array}{r} 3x^4-2x^3y+5x^2 \\ +6x^3y-4x^2y^2+10xy \\ -9x^2+6xy-15 \\ \hline \end{array}$$

$$\hline 3x^4+4x^3y-4x^2-4x^2y^2+16xy-15$$

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$.
Product $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.
Product x^4+y^4+
5. Multiply x^2+xy+y^2 by x^2-xy+y^2 .
Product $x^4+x^2y^2+y^4$
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
Product $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$

D I V I S I O N.

C A S E I.

When the divisor is a simple quantity.

R U L E.

1. Place the dividend above a small line, and the divisor under it, in the manner of a vulgar fraction.

2. Expunge those letters that are common to both the dividend and divisor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the result will be the quotient required.

Note. Like signs make +, and unlike signs —, the same as in multiplication.

E X A M P L E S:

$$\frac{a}{a}=1; \quad \frac{8bc}{2b}=4c; \quad \frac{abc}{bcd}=\frac{a}{d}; \quad \frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d};$$

$$\frac{ab+b^2}{2b}=\frac{a+b}{2}; \quad \frac{12xy}{6x^2}=\frac{2y}{x}; \quad \frac{30ax-54ay}{12ab}=\frac{5x-9y}{2b};$$

$$\frac{10x^2y-15y^2-5y}{5y}=2x^2-3y-1.$$

1. Divide $18x^2$ by $9x$. Quotient $2x$.
2. Divide $10x^2y^2$ by $-5x^2y$. Quotient $-2y$.
3. Divide $-9ax^2y^2$ by $9x^2y$. Quotient $-ay$.
4. Divide $-8x^2$ by $-2x$. Quotient $+4x$.
5. Divide $a+3ax-x^2$ by a . Quotient $1+3x-\frac{x^2}{a}$.
6. Divide $3a^2-15+6a+3b$ by $3a$.
Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

C A S E II.

When the divisor and dividend are both compound quantities.

R U L E.

1. Range the terms of both the quantities according to the dimensions of some letter in them; so that the first term may have the highest power of that letter, and the second term the next highest power; and so on.
2. Divide the first term of the dividend by the first term of the divisor, and place the result in the quotient.

3. Multiply the whole divisor by the quotient term last found, and subtract the result from the dividend.

4. To this remainder bring down the next term of the dividend, and divide as before; and so on, as in common arithmetic.

Note, Like signs produce +, and unlike signs —.

EXAMPLES:

$$(a+x) a^3 + 5a^2x + 5ax^2 + x^3 \div (a^2 + 4ax + x^2)$$

$$\underline{4a^2x + 5ax^2}$$

$$\underline{4a^2x + 4ax^2}$$

$$ax^2 + x^3$$

$$\underline{ax^2 + x^3}$$

*

$$(x-3)x^3 - 9x^2 + 27x - 27 \div (x^2 - 6x + 9)$$

$$\underline{x^3 - 3x^2}$$

$$\underline{-6x^2 + 27x}$$

$$\underline{-6x^2 + 18x}$$

$$9x - 27$$

$$\underline{9x - 27}$$

*

by

$$a-x) a^3-x^3(a^2+ax+x^2)$$

$$\underline{a^3-a^2x}$$

$$a^2x-x^3$$

$$\underline{a^2x-ax^2}$$

$$ax^2-x^3$$

$$\underline{ax^2-x^3}$$

*

$$b-y) b^4-y^4(b^3+b^2y+by^2+y^3)$$

$$\underline{b^4-b^3y}$$

$$b^3y-y^4$$

$$\underline{b^3y-b^2y^2}$$

$$b^2y^2-y^4$$

$$\underline{b^2y^2-by^3}$$

$$by^3-y^4$$

$$\underline{by^3-y^4}$$

*

EXAMPLES FOR PRACTICE.

1. Divide $a^3+2ax+x^2$ by $a+x$. Quotient $a+x$

2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.

Quotient $a^2-2ay+y^2$

3. Divide 1 by $1-x$. Quotient $1+x+x^2+x^3$ &c.

4. Divide $6x^4-96$ by $3x-6$.

Quotient $2x^3+4x^2+8x+16$

5. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. Quotient $a^3-3a^2x+3ax^2-x^3$

ALGEBRAIC FRACTIONS.

PROBLEM I.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; and the denominator being placed under this sum will give the improper fraction required.

EXAMPLES:

$$3\frac{5}{7} = \frac{3 \times 7 + 5}{7} = \frac{26}{7}; \quad x + \frac{x^2}{a} = \frac{ax + x^2}{a};$$

$$a - \frac{b}{c} = \frac{ac - b}{c}; \quad 5\frac{2}{3} = \frac{5 \times 3 + 2}{3} = \frac{17}{3};$$

$$1 - \frac{2x}{a} = \frac{a - 2x}{a}; \quad a - x + \frac{a^2 - ax}{x} = \frac{a^2 - x^2}{x}.$$

Let the following mixed quantities be reduced to improper fractions:

$$\frac{86}{7}; \quad a^2 - \frac{2a}{b}; \quad 1 + 2x - \frac{x-3}{5x}; \quad b + \frac{1-x+a}{c};$$

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part, and place the remainder over the denominator for the fractional part, and it will be the mixed quantity required.

EXAMPLES:

$$\frac{17}{5} = 3\frac{2}{5}; \quad \frac{ax+a^2}{x} = a + \frac{a^2}{x}; \quad \frac{ay+2y^2}{a+y} = y + \frac{y^2}{a+y};$$

$$\frac{25}{4} = 6\frac{1}{4}; \quad \frac{ab-a^2}{b} = a - \frac{a^2}{b}; \quad \frac{a^2+x^2}{a-x} = a+x + \frac{2x^2}{a-x}.$$

Let the following improper fractions be reduced to mixed quantities.

$$\frac{35}{8}; \quad \frac{28}{4}; \quad \frac{3ab-b^2}{a}; \quad \frac{2x^2y}{2x}; \quad \frac{b-2x^2}{b+x};$$

PROBLEM III.

To reduce fractions of different denominators, to those of the same value that shall have a common denominator.

R U L E.

Multiply every numerator separately into all the denominators but its own for the new numerators, and all the denominators together for the common denominator required.

E X A M P L E S :

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to fractions of equal values that shall have a common denominator.

$$\begin{array}{rcl} a \times c & = & ac \\ b \times b & = & b^2 \end{array}$$

$$b \times c = bc$$

$$\frac{ac}{bc} \text{ and } \frac{b^2}{bc} = \text{fractions required.}$$

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{c}{d}$ to equivalent fractions, having a common denominator.

$$\begin{array}{rcl} a \times c \times d & = & acd \\ b \times b \times d & = & b^2d \\ c \times b \times c & = & c^2b \end{array}$$

$$b \times c \times d = bcd$$

$$\frac{acd}{bcd}, \frac{b^2d}{bcd}, \text{ and } \frac{c^2b}{bcd} = \text{fractions required.}$$

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Ans.* $\frac{2xc}{ac}$ and $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator. *Ans.* $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d to a common denominator. *Ans.* $\frac{9xc}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6adc}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{2x}{a}$ to a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$ and $\frac{12a^2 + 24x}{12a}$.

PROBLEM IV.

To find the greatest common measure of a fraction.

RULE.

1. Range the quantities according to the dimensions of some letter, as is shewn in division.

2. Divide the greater term by the less, and the last divisor by the last remainder, and so on till nothing remains; and the divisor last used will be the common measure required.

Note. All the letters or figures that are common to each divisor, must be thrown out of them before they are used in the operation.

EXAMPLES:

1. To find the greatest common measure of

$$\frac{cx+x^2}{ca^2+a^2x}$$

$$\begin{array}{r} (cx+x^2)ca^2+a^2x \\ \text{or } c+x)ca^2+a^2x(a^2 \\ \hline ca^2+a^2x \end{array}$$

*

Therefore the greatest common measure is $c+x$.

2. To find the greatest common measure of

$$\frac{x^3-b^2x}{x^2+2bx+b^2}$$

$$\begin{array}{r} x^3+2bx+b^2)x^3-b^2x(x \\ \hline x^3+2bx^2+b^2x \\ \hline -2bx^2-2b^2x)x^2+2bx+b^2 \\ \text{or } x+b)x^2+2bx+b^2(x+b \\ \hline x^2+bx \\ \hline \end{array}$$

$$\begin{array}{r} bx+b^2 \\ \hline bx+b^2 \\ \hline \end{array}$$

Therefore $x+b$ is the greatest common divisor.

3. To find the greatest common divisor of

$$\frac{x^4-b^4}{x^5-b^2x^3} \quad \text{Ans. } x^2-b^2.$$

4. To find the greatest common measure of

$$\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4} \quad \text{Ans. } a+b.$$

PROBLEM V.

To reduce a fraction to its lowest terms.

RULE.

1. Find the greatest common measure, as in the last problem.
2. Divide both the terms of the fraction by the common measure thus found, and it will reduce the fraction as required.

EXAMPLES:

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms.

$$\begin{array}{r} cx+x^2)ca^2+a^2x \\ \text{or } c+x)ca^2+a^2x(a^2 \\ \hline ca^2+a^2x \end{array}$$

Therefore $c+x$ is the greatest common measure,

$$\text{and } c+x) \frac{cx+x^2}{ca^2+a^2x} = \frac{x}{a^2} = \text{fraction required.}$$

2. Having $\frac{x^3-b^2x}{x^2+2bx+b^2}$ given, it is required to reduce it to its least terms.

$$\begin{array}{r} x^2+2bx+b^2)x^3-b^2x(x \\ \hline x^3+2bx^2+b^2x \end{array}$$

$$\begin{array}{r} -2bx^2-2b^2x)x^2+2bx+b^2 \\ \text{or } x+b)x^2+2bx+b^2(x+b \\ \hline x^2+bx \end{array}$$

$$\begin{array}{r} bx+b^2 \\ bx+b^2 \\ \hline \end{array}$$

*

Therefore $x+b$ is the greatest common measure,
 and $x+b \overline{) \frac{x^3-b^2x}{x^2+2bx+b^2}} = \frac{x^2-bx}{a+b}$ = fraction required.

3. To reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms.

$$\text{Ans. } \frac{x^2+b^2}{x^3}$$

4. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

$$\text{Ans. } \frac{a^2+x^2}{a-x}$$

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator, as in problem the third.
2. Add all the numerators together, and under their sum write the common denominator, and it will give the sum of the fractions required.

EXAMPLES:

1. Having $\frac{x}{2}$ and $\frac{x}{3}$ given, to find their sum.

$$x \times 3 = 3x$$

$$x \times 2 = 2x$$

$$2 \times 3 = 6$$

$$\frac{3x}{6} + \frac{2x}{6} = \frac{5x}{6} = \text{sum required.}$$

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2. Having $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ given, to find their sum.

$$a \times d \times f = adf$$

$$c \times b \times f = cbf$$

$$e \times b \times d = ebd$$

$$b \times d \times f = bdf$$

$$\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf} = \text{sum required.}$$

3. Add $a - \frac{3x^2}{b}$ and $b + \frac{x-2b}{c}$ together.

$$\frac{3x^2 \times c}{x-2b \times b} = \frac{3cx^2}{bx-2b^2}$$

$$b \times c = bc$$

$$a - \frac{3cx^2}{bc}$$

$$b + \frac{bx-2b^2}{bc}$$

$$a + b + \frac{bx-3cx^2-2b^2}{bc} = \text{sum required.}$$

4. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together. Sum $\frac{15x+2bx}{10b}$

5. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. Sum $x + \frac{x}{12}$

6. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. Sum $\frac{19x-14}{21}$

7. Add $x + \frac{x-2}{3}$ and $3x + \frac{2x-3}{4}$ together.

$$\text{Sum. } 4x + \frac{10x-17}{12}$$

PROBLEM VII.

To subtract one fractional quantity from another.

RULE.

1. Reduce the fractions to a common denominator, as in addition.
2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES:

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

$$\begin{array}{r} x \times 11 = 11x \\ 2x \times 3 = 6x \end{array}$$

$$3 \times 11 = 33$$

$$\frac{11x}{33} - \frac{6x}{33} = \frac{5x}{33} \text{ difference required.}$$

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

$$\begin{array}{r} \overline{x-a} \times 5c = 5cx - 5ac \\ 2a-4x \times 3b = 6ab - 12bx \end{array}$$

$$3b \times 5c = 15bc$$

$$\frac{5cx-5ac}{15bc} - \frac{6ab-12bx}{15bc} = \frac{5cx-5ac-6ab+12bx}{15bc} = \text{difference required.}$$

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3. From $\frac{3x}{7}$ take $\frac{2x}{9}$. *Difference* $\frac{13x}{63}$.

4. From $\frac{x+a}{b}$ take $\frac{c}{d}$. *Difference* $\frac{dx+ad-bc}{bd}$.

5. From $\frac{3x+a}{5b}$ take $\frac{2x+7}{8}$.
Difference $\frac{24x+8a-10bx-35b}{40b}$.

6. From $3x+\frac{x^2}{b}$ take $x-\frac{x-a}{c}$.
Difference $2x+\frac{cx^2-bx+ab}{bc}$.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator, and it will give the product required.

EXAMPLES:

1. Find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

$$\left. \begin{array}{l} x \times 2x \\ 6 \times 9 \end{array} \right\} = \frac{2x^2}{54} = \frac{x^2}{27} = \text{product required.}$$

D

2. Find the product of $\frac{x}{2}$, $\frac{4x}{5}$ and $\frac{10x}{21}$.

$$\frac{x \times 4x \times 10x}{2 \times 5 \times 21} \Bigg\} = \frac{40x^3}{210} = \frac{4x^3}{21} = \text{product required.}$$

3. Find the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

$$\frac{x \times x+a}{a \times a+c} \Bigg\} = \frac{x^2+ax}{a^2+ac} = \text{product required.}$$

4. Find the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. Product $\frac{9ax}{2b}$.

5. Find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. Product $\frac{3x^3}{5a}$.

6. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$. Product $9xa$.

7. Find the product of $b + \frac{bx}{a}$ and $\frac{a}{x}$.

$$\text{Product } \frac{ab+bx}{x}$$

8. Find the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

$$\text{Product } \frac{x^4-b^4}{cb^2-bc^2}$$

PROBLEM IX.

To divide one fractional quantity by another.

RULE.

Invert the divisor, and proceed as in multiplication.

In-
posed
cube,

FRACTIONS.

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EXAMPLES:

1. Find the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

$$\frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\frac{1}{2} = \text{quotient required.}$$

2. Find the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

$$\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc} = \text{quotient required.}$$

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$$\frac{x+a}{2x-2b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{2x^2-2b^2} = \text{quotient required.}$$

4. Divide $\frac{5x}{3}$ by $\frac{2a}{3b}$.

$$\text{Quotient } \frac{5bx}{a} \quad \times$$

5. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$.

$$\text{Quotient } \frac{x-b}{6c^2x}$$

6. Divide $\frac{x^2-b^2}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

$$\text{Quotient } x + \frac{b^2}{x}$$

INVOLUTION.

Involution is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c. of any given quantity.

R U L E.

Multiply the quantity into itself as often as is denoted by the index, and the last product will be the power required. Or

Multiply the index of the quantity by the index of the power, and the result will be the same as before.

Note, When the sign of the root is + all the powers of it will be +; and when the sign is - all the odd powers will be -, and all the even powers +.

E X A M P L E S :

$$a, \text{ root } \left\{ \begin{array}{l} a^2 = \text{square} \\ a^3 = \text{cube} \\ a^4 = 4^{\text{th}} \text{ power} \\ a^5 = 5^{\text{th}} \text{ power} \\ \text{\&c.} \end{array} \right.$$

$$a^2, \text{ root } \left\{ \begin{array}{l} a^4 = \text{square} \\ a^6 = \text{cube} \\ a^8 = 4^{\text{th}} \text{ power} \\ a^{10} = 5^{\text{th}} \text{ power} \\ \text{\&c.} \end{array} \right.$$

$$-3a, \text{ root } \left\{ \begin{array}{l} + 9a^2 = \text{square} \\ - 27a^3 = \text{cube} \\ + 81a^4 = 4^{\text{th}} \text{ power} \\ - 243a^5 = 5^{\text{th}} \text{ power} \end{array} \right.$$

$$-2ax^2, \text{ root } \left\{ \begin{array}{l} + 4a^2x^4 = \text{square} \\ - 8a^3x^6 = \text{cube} \\ + 16a^4x^8 = 4^{\text{th}} \text{ power} \\ - 32a^5x^{10} = 5^{\text{th}} \text{ power} \end{array} \right.$$

$$\frac{x}{a}, \text{ root } \left\{ \begin{array}{l} \frac{x^2}{a^2} = \text{square} \\ \frac{x^3}{a^3} = \text{cube} \\ \frac{x^4}{a^4} = \text{biquadrate} \end{array} \right.$$

$$\left. \begin{array}{l} -\frac{2ax^2}{3b}, \text{ root} \\ -\frac{2ax^2}{3b}, \text{ root} \end{array} \right\} \begin{array}{l} + \frac{4a^2x^4}{9b^2} = \text{square} \\ - \frac{8a^3x^6}{27b^3} = \text{cube} \\ + \frac{16a^4x^8}{81b^4} = 4^{\text{th}} \text{ power} \end{array}$$

$$x + a = \text{root}$$

$$x + a$$

$$\begin{array}{r} x^2 + ax \\ + ax + a^2 \\ \hline \end{array}$$

$$\begin{array}{r} x^2 + 2ax + a^2 = \text{square} \\ x + a \\ \hline \end{array}$$

$$\begin{array}{r} x^3 + 2ax^2 + a^2x \\ + ax^2 + 2a^2x + a^3 \\ \hline \end{array}$$

$$\begin{array}{r} x^3 + 3ax^2 + 3a^2x + a^3 = \text{cube} \\ x + a \\ \hline \end{array}$$

$$\begin{array}{r} x^4 + 3ax^3 + 3a^2x^2 + a^3x \\ + ax^3 + 3a^2x^2 + 3a^3x + a^4 \\ \hline \end{array}$$

$$x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4 = 4^{\text{th}} \text{ power.}$$

EXAMPLES FOR PRACTICE.

1. Required the cube or third power of $2a^2$.
Ans. $8a^6$.
2. Required the 4th power of $2a^2x$. *Ans.* $16a^8x^4$.
3. To find the 3d power of $-8x^2y^3$.
Ans. $512x^6y^9$.

4. To find the biquadrate of $-\frac{2a^2x}{3b^2}$.

Ans. $\frac{16a^8x^4}{81b^8}$.

5. Required the 5th power of $a-x$.

Ans. $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$.

SIR ISAAC NEWTON'S RULE for raising a binomial or residual quantity to any power whatever.

1. To find the terms without the co-efficients. The index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. To find the unciæ or co-efficients. The first is always 1, and the second is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

Note, The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term be -, then all the odd terms will be +, and the even terms -.

EXAMPLES:

1. Let $a+x$ be involved to the fifth power.

The terms without the co-efficients will be

$a^5, a^4x, a^3x^2, a^2x^3, ax^4, x^5,$

EVOLUTION.

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and the co-efficients will be

$$1, 5, \frac{5 \times 4}{2}, \frac{10 \times 3}{3}, \frac{10 \times 2}{4}, \frac{5 \times 1}{5};$$

or 1, 5, 10, 10, 5, 1,

And therefore the 5th power is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

2. Let $x-a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6, x^5a, x^4a^2, x^3a^3, x^2a^4, xa^5, a^6;$$

and the co-efficients will be

$$1, 6, \frac{6 \times 5}{2}, \frac{15 \times 4}{3}, \frac{20 \times 3}{4}, \frac{15 \times 2}{5}, \frac{6 \times 1}{6},$$

or 1, 6, 15, 20, 15, 6, 1,

And therefore the sixth power of $x-a$ is

$$x^6 - 6x^5a + 15x^4a^2 - 20x^3a^3 + 15x^2a^4 - 6xa^5 + a^6.$$

3. Find the 4th power of $x-a$.

$$\text{Ans. } x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4.$$

4. Find the 7th power of $x+a$.

$$\text{Ans. } x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7.$$

EVOLUTION.

Evolution is the reverse of involution, and teaches to find the roots of any given powers.

CASE I.

To find the roots of simple quantities.

RULE.

Extract the root of the co-efficient for the numerical part, and divide the index of the letters by the index of the power, and it will give the root required.

EXAMPLES:

1. The square root of $9x^2 = 3x^{\frac{2}{2}} = 3x$.
2. The cube root of $8x^3 = 2x^{\frac{3}{3}} = 2x$.
3. The square root of $3a^2x^6 = a^{\frac{2}{2}}x^{\frac{6}{2}}\sqrt{3} = ax^3\sqrt{3}$.
4. The cube root of $-125a^3x^6 = 5a^{\frac{3}{3}}x^{\frac{6}{3}} = 5ax^2$.
5. The biquadrate root of $16a^4x^8 = 2a^{\frac{4}{4}}x^{\frac{8}{4}} = 2ax^2$.

CASE II.

To find the square root of a compound quantity.

RULE.

1. Range the quantities according to the dimensions of some letter, and set the root of the first term in the quotient.
2. Subtract the square of the root, thus found, from the first term, and bring down the two next terms to the remainder for a dividend.
3. Divide the dividend by double the root, and set the result in the quotient. *term last put into the*
4. Multiply the divisor and quotient by the term last put in the quotient, and subtract the product from the dividend, and so on, as in common arithmetic.

EXAMPLES:

1. Extract the square root of $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.

$$\begin{array}{r} 4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4 \\ 4a^4 \end{array}$$

$$\begin{array}{r} 4a^2 + 3ax \quad 12a^3x + 13a^2x^2 \\ 12a^3x + 9a^2x^2 \end{array}$$

$$\begin{array}{r} 4a^2 + 6ax + x^2 \quad 4a^2x^2 + 6ax^3 + x^4 \\ 4a^2x^2 + 6ax^3 + x^4 \end{array}$$

*

2. Extract the square root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

$$\begin{array}{r} x^4 - 4x^3 + 6x^2 - 4x + 1 \\ x^4 \end{array}$$

$$\begin{array}{r} 2x^2 - 2x \quad -4x^3 + 6x^2 \\ -4x^3 + 4x^2 \end{array}$$

$$\begin{array}{r} 2x^2 - 4x + 1 \quad 2x^2 - 4x + 1 \\ 2x^2 - 4x + 1 \end{array}$$

*

3. Required the square root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.

$$\text{Ans. } a^2 + 2ax + x^2.$$

4. Required the square root of $x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$.

$$\text{Ans. } x^2 - x + \frac{1}{4}.$$

5. Required the square root of $a^2 + x^2$.

$$\text{Ans. } a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \&c.$$

CASE III.

To find the roots of powers in general.

R U L E.

1. Find the root of the first term, and place it in the quotient.
2. Subtract the power, and bring down the second term for a dividend.
3. Involve the root, last found, to the next lowest power, and multiply it by the index of the given power for a divisor.
4. Divide the dividend by the divisor, and the quotient will be the next term of the root.
5. Involve the whole root, and subtract and divide as before; and so on till the whole is finished.

E X A M P L E S :

1. Required the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$$\begin{array}{r} a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \div (a^2 - ax + x^2) \\ \underline{a^4} \end{array}$$

$$2a^2) - 2a^3x$$

$$\underline{a^4 - 2a^3x + a^2x^2}$$

$$2a^2) 2a^2x^2$$

$$\underline{a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4}$$

*

2. Extract the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$$\begin{array}{r}
 x^6 + 6x^5 - 40x^3 + 96x - 64 \div (x^2 + 2x - 4) \\
 \underline{3x^4) 6x^5} \\
 \underline{x^6 + 6x^5 + 12x^4 + 8x^3} \\
 3x^4 - 12x^3 \\
 \underline{ 3x^4 - 12x^3} \\
 x^6 + 6x^5 - 40x^3 + 96x - 64
 \end{array}$$

3. Required the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$. *Ans.* $a + b + c$.

4. Required the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$. *Ans.* $x^2 - 2x + 1$.

5. Required the biquadrate root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$. *Ans.* $2a - 3x$.

S U R D S.

Surds are such quantities as have no exact root, and are usually expressed by fractional indices: thus, the square root of 2, and the cube root of 3, &c. cannot be exactly determined, but may be denoted by $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ &c.

P R O B L E M I.

To reduce a rational quantity to the form of a Surd.

R U L E.

Multiply the index of the rational quantity by the index of the surd, and over this new quantity place the radical sign, and it will be of the form required.

E X A M P L E S :

1. 3 reduced to the form of $\sqrt{7}$ is $\sqrt{3^{1 \times 2}} = \sqrt{3^2} = \sqrt{9}$.

2. x reduced to the form of $\sqrt[3]{a}$ is $\sqrt[3]{x^{1 \times 3}} = \sqrt[3]{x^3}$.

3. $a+b$ reduced to the form of $\sqrt{cx}$ is $\sqrt{a+b}^2 = \sqrt{a^2 + 2ab + b^2}$.

4. $\frac{a}{b\sqrt{x}}$ reduced to the form of $\sqrt{c}$ is $\sqrt{\frac{a^2}{b^2x}}$.

P R O B L E M II.

To reduce quantities of different indices to other equivalent ones, that shall have a common index.

R U L E.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities.

2. Over the said quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

E X A M P L E S :

1. Reduce $15^{\frac{1}{2}}$ and $9^{\frac{1}{3}}$ to equivalent quantities having the common index $\frac{1}{6}$.

$$\frac{2}{4} \div \frac{1}{4} = \frac{2}{4} \times \frac{4}{1} = \frac{2}{1} = \frac{2}{1} = 1^{\text{st}} \text{ index.}$$

$$\frac{1}{6} \div \frac{1}{3} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2} = \frac{1}{2} = 2^{\text{d}} \text{ index}$$

Therefore $15^{\frac{1}{2}}$ and $9^{\frac{1}{3}}$ = quantities required.

2. Reduce a^2 and $x^{\frac{1}{2}}$ to the same common index $\frac{1}{6}$.

$$\frac{2}{1} \div \frac{1}{6} = \frac{2}{1} \times \frac{6}{1} = \frac{6}{1} = \frac{6}{1} = 1^{\text{st}} \text{ index}$$

$$\frac{1}{4} \div \frac{1}{3} = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4} = \frac{3}{4} = 2^{\text{d}} \text{ index}$$

Therefore a^6 and $x^{\frac{3}{4}}$ = quantities required.

3. Reduce $3^{\frac{2}{3}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Ans. $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{12}$.

Ans. $a^{\frac{4}{12}}$ and $b^{\frac{3}{12}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the same radical sign.

Ans. ${}^{nm}\sqrt{a^m}$ and ${}^{nm}\sqrt{b^n}$.

PROBLEM III.

To reduce surds to their most simple terms.

R U L E.

Find the root of the greatest power contained in the given surd, and set it before the remaining quantities, with the proper radical sign between them.

EXAMPLES:

1. $\sqrt{48} = \sqrt{3 \times 16} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}.$

2. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}.$

E

$$3. \sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27} \times \sqrt[3]{3} = 3\sqrt[3]{3}.$$

$$4. \sqrt{49a^2x} = \sqrt{49a^2} \times \sqrt{x} = \sqrt{49a^2} \times \sqrt{x} = 7a\sqrt{x}.$$

$$5. \sqrt{bx+x^2}^{\frac{1}{2}} = x^{\frac{1}{2}} \times \sqrt{b+x}^{\frac{1}{2}}; \quad \sqrt{a^3x-a^2x^2}^{\frac{1}{2}} = a \times \sqrt{ax-x^2}^{\frac{1}{2}}.$$

$$6. \sqrt[3]{\frac{64a^4x^3}{27a-27x}} = \frac{4ax}{3} \times \sqrt[3]{\frac{a}{a-x}}^{\frac{1}{3}}.$$

PROBLEM IV.

To add surd quantities together.

RULE.

1. Reduce those quantities that have unlike indices to equivalent ones, having a common index.
2. Bring all fractions to a common denominator, and reduce the quantities to their simplest terms, as in the last problem.
3. Then, if the surd part be the same in them all, annex it to the sum of the rational parts, with the sign of multiplication, and it will give the total sum required.

If the surd part be not the same in all the quantities, they can only be added by the signs + and -.

EXAMPLES:

$$1. 2\sqrt{5} + 7\sqrt{5} = 2 + 7 \times \sqrt{5} = 9\sqrt{5}.$$

$$2. \sqrt{27} + \sqrt{48} = 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}.$$

$$3. \sqrt[3]{500} + \sqrt[3]{108} = \sqrt[3]{125 \times 4} + \sqrt[3]{27 \times 4} = 5 \times 4^{\frac{1}{3}} + 3 \times 4^{\frac{1}{3}} = 8 \times 4^{\frac{1}{3}}.$$

$$4. \sqrt{27a^4x} + \sqrt{3a^2x^3} = \sqrt{9a^4 \times 3x} + \sqrt{a^2x^2 \times 3x} = 3a^2\sqrt{3x} + ax\sqrt{3x} = 3a + x \times a \times \sqrt{3x}.$$

$$5. \sqrt{\frac{24}{25}} + \sqrt{\frac{2}{3}} = \sqrt{\frac{72}{75}} + \sqrt{\frac{50}{75}} = \sqrt{\frac{36 \times 2}{75}} + \sqrt{\frac{25 \times 2}{75}} = 6\sqrt{\frac{2}{75}} + 5\sqrt{\frac{2}{75}} = 11\sqrt{\frac{2}{75}}.$$

$$6. \sqrt[3]{\frac{1}{32}} + \sqrt[3]{\frac{64}{108}} = \sqrt[3]{\frac{1}{8 \times 4}} + \sqrt[3]{\frac{64 \times 1}{27 \times 4}} = \frac{1}{2} \times \sqrt[3]{\frac{1}{4}} + \frac{4}{3} \times \sqrt[3]{\frac{1}{4}} = \frac{1}{2} + \frac{4}{3} \times \sqrt[3]{\frac{1}{4}} = \frac{11}{6} \times \sqrt[3]{\frac{1}{4}}.$$

$$7. \sqrt{\frac{a}{b}} + \sqrt{\frac{b^3}{a^3}} = \sqrt{\frac{a^4}{ba^3}} + \sqrt{\frac{b^4}{ba^3}} = \frac{a^2}{a} \times \sqrt{\frac{1}{ba}} + \frac{b^2}{a} \times \sqrt{\frac{1}{ba}} = \frac{a^2 + b^2}{a} \times \sqrt{\frac{1}{ba}} = \frac{a^2 + b^2}{a\sqrt{ba}}.$$

PROBLEM V.

To subtract surd quantities.

RULE.

Prepare the quantities as in the last rule, and annex the difference of the rational parts to the common surd, with the sign of multiplication.

EXAMPLES:

$$1. 3\sqrt{18} - \sqrt{18} = 3 - 1 \times \sqrt{18} = 2\sqrt{18}.$$

$$2. \ 2\sqrt{50} - \sqrt{18} = 2\sqrt{25 \times 2} - \sqrt{9 \times 2} = 2 \times 5 \times \sqrt{2} - 1 \times 3 \times \sqrt{2} = 7\sqrt{2}.$$

$$3. \ \sqrt[3]{192} - \sqrt[3]{24} = \sqrt[3]{64 \times 3} - \sqrt[3]{8 \times 3} = 4 \times \sqrt[3]{3} - 2 \times \sqrt[3]{3} = 4 - 2 \times \sqrt[3]{3} = 2 \times \sqrt[3]{3}.$$

$$4. \ \sqrt{80a^4x} - \sqrt{20a^2x^3} = \sqrt{16a^4 \times 5x} - \sqrt{4a^2x^2 \times 5x} = 4a^2\sqrt{5x} - 2ax\sqrt{5x} = 4a^2 - 2ax \times \sqrt{5x}.$$

$$5. \ \sqrt{\frac{24}{25}} - \sqrt{\frac{2}{3}} = \sqrt{\frac{72}{75}} - \sqrt{\frac{50}{75}} = \sqrt{\frac{36 \times 2}{75}} - \sqrt{\frac{25 \times 2}{75}} = 6 \times \sqrt{\frac{2}{75}} - 5 \times \sqrt{\frac{2}{75}} = 6 - 5 \times \sqrt{\frac{2}{75}} = \sqrt{\frac{2}{75}}.$$

$$6. \ \sqrt[3]{\frac{16}{27}} - \sqrt[3]{\frac{1}{4}} = \frac{4}{3} \times \sqrt[3]{\frac{1}{4}} - \sqrt[3]{\frac{1}{4}} = \frac{4}{3} - \frac{3}{3} \times \sqrt[3]{\frac{1}{4}} = \frac{1}{3} \times \sqrt[3]{\frac{1}{4}}.$$

$$7. \ \sqrt{\frac{b}{c}} - \sqrt{\frac{c^2}{b^3}} = \sqrt{\frac{b^4}{cb^3}} - \sqrt{\frac{c^4}{cb^3}} = \frac{b^2}{b} \times \sqrt{\frac{1}{cb}} - \frac{c^2}{b} \times \sqrt{\frac{1}{cb}} = \frac{b^2 - c^2}{b} \times \sqrt{\frac{1}{cb}} = \frac{b^2 - c^2}{b \times cb^{\frac{1}{2}}}.$$

PROBLEM VI.

To multiply surd quantities together.

RULE.

Reduce the surds to the same index ; and then the product of the rational quantities being annexed to

the product of the surds, will give the whole product required.

EXAMPLES.

$$1. \sqrt{10} \times \sqrt{5} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}.$$

$$2. 3\sqrt{8} \times 2\sqrt{6} = 3 \times 2 \sqrt{8 \times 6} = 6\sqrt{48} = 6\sqrt{16 \times 3} = 4 \times 6 \times \sqrt{3} = 24\sqrt{3}.$$

$$3. \frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{2}{3}\sqrt{\frac{8}{64}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{6}{12}\sqrt{\frac{56}{640}} = \frac{1}{2}\sqrt{\frac{4 \times 14}{64 \times 10}} = \frac{1}{2} \times \frac{2}{8} \times \sqrt{\frac{14}{10}} = \frac{2}{16}\sqrt{1\frac{4}{5}} = \frac{1}{8}\sqrt{1\frac{2}{5}}.$$

$$4. \frac{1}{3} \times^3 \sqrt{\frac{3}{2}} \times \frac{1}{5} \times^3 \sqrt{\frac{1}{4}} = \frac{1}{3} \times^3 \sqrt{\frac{3}{2} \times \frac{1}{4}} = \frac{1}{3} \times^3 \sqrt{\frac{3 \times 3}{2 \times 12}} = \frac{1}{15} \times^3 \sqrt{\frac{9}{24}} = \frac{1}{15} \times^3 \sqrt{\frac{3}{8}} = \frac{1}{15} \times \frac{1}{2} \times^3 \sqrt{\frac{1}{3}} = \frac{1}{30} \times^3 \sqrt{\frac{1}{3}}.$$

$$5. a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3}{6}} \times a^{\frac{2}{6}} = a^{\frac{5}{6}} = \sqrt[6]{a^5}.$$

$$6. 5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 6x^{\frac{2}{3}}y^{\frac{3}{4}} = 5x^{\frac{6}{12}}y^{\frac{4}{12}} \times 6x^{\frac{8}{12}}y^{\frac{9}{12}} = 30x^{\frac{14}{12}}y^{\frac{13}{12}} = 30 \times x^{1\frac{1}{3}}y^{1\frac{1}{4}}.$$

$$7. \sqrt[3]{x+y}^{\frac{1}{2}} \times \sqrt[3]{x+y}^{\frac{1}{3}} = \sqrt[3]{x+y}^{\frac{3}{6}} \times \sqrt[3]{x+y}^{\frac{2}{6}} = \sqrt[3]{x+y}^{\frac{5}{6}} = \sqrt[6]{(x+y)^5}.$$

$$8. \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} \times \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} \times \frac{x^{\frac{2}{2}}}{y^{\frac{2}{2}}} = \frac{x^{\frac{3}{2}}}{y^{\frac{3}{2}}} \times \frac{x^{\frac{2}{2}}}{y^{\frac{2}{2}}} = \frac{x^{\frac{5}{2}}}{y^{\frac{5}{2}}}.$$

$$9. x^{\frac{1}{n}} \times y^{\frac{1}{m}} = x^{\frac{m}{nm}} \times y^{\frac{n}{nm}} = x^{\frac{1}{nm}} \times y^{\frac{1}{nm}} = x^m y^n \frac{1}{nm}.$$

$$10. x + \sqrt{y} \times x - \sqrt{y} = x^2 - y.$$

$$11. x - x\sqrt{a} \times y - y\sqrt{a} = xy - 2xy\sqrt{a} + axy.$$

$$12. \sqrt{a + \sqrt{b} - \sqrt{3}} \times \sqrt{a - \sqrt{b} - \sqrt{3}} \\ = \sqrt{a^2 - b + \sqrt{3}}.$$

PROBLEM VII.

To divide one surd quantity by another.

RULE.

Reduce the surds to the same index; and then the quotient of the rational quantities being annexed to the quotient of the surds, will give the whole quotient required.

EXAMPLES:

$$1. \sqrt{375} \div \sqrt{5} = \sqrt{\frac{375}{5}} = \sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}.$$

$$2. 8\sqrt{108} \div 2\sqrt{6} = \frac{8}{2} \sqrt{\frac{108}{6}} = 4\sqrt{18} = 4\sqrt{9 \times 2} \\ = 12\sqrt{2}.$$

$$3. \frac{3}{4} \sqrt{\frac{1}{135}} \div \frac{2}{3} \sqrt{\frac{1}{5}} = \frac{3}{4} \times \frac{3}{2} \times \sqrt{\frac{1}{135} \times \frac{5}{1}} \\ = \frac{9}{8} \sqrt{\frac{5}{135}} = \frac{9}{8} \sqrt{\frac{1}{9 \times 3}} = \frac{3}{8} \sqrt{\frac{1}{3}}.$$

$$4. \frac{5}{7} \times^3 \sqrt{\frac{8}{189}} \div \frac{2}{5} \times^3 \sqrt{\frac{8}{7}} = \frac{5}{7} \times \frac{5}{2} \times \\ \frac{8}{189} \times \frac{7}{8} \sqrt[3]{\frac{1}{42}} = \frac{25}{42} \times \sqrt[3]{\frac{1}{42}}.$$

$$5. x^{\frac{1}{2}} \div y^{\frac{1}{3}} = x^{\frac{3}{6}} \div y^{\frac{2}{6}} = \sqrt[6]{x^3 \div y^2} = \sqrt[6]{\frac{x^3}{y^2}}.$$

$$6. \frac{2}{5} x^{\frac{1}{2}} y^{\frac{1}{3}} \div \frac{3}{4} x^{\frac{1}{3}} y^{\frac{1}{4}} = \frac{2}{5} x^{\frac{2}{6}} y^{\frac{2}{6}} \div \frac{3}{4} x^{\frac{2}{6}} y^{\frac{2}{6}} = \frac{2}{5} \times \\ \frac{4}{3} \times x^{\frac{2}{6}} y^{\frac{2}{6}} = \frac{8}{15} \times \sqrt[6]{xy}.$$

$$7. \sqrt{x+y}^{\frac{1}{2}} \div \sqrt{x+y}^{\frac{1}{3}} = \sqrt{x+y}^{\frac{3}{6}} \div \sqrt{x+y}^{\frac{2}{6}} = \sqrt{x+y}^{\frac{1}{6}}.$$

$$8. \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \div \frac{y^{\frac{1}{4}}}{x^{\frac{1}{5}}} = \frac{x^{\frac{1}{2}}}{y^{\frac{1}{6}}} \times \frac{x^{\frac{1}{5}}}{y^{\frac{1}{4}}} = \frac{x^{\frac{5}{10}}}{y^{\frac{5}{10}}} = \sqrt[10]{\frac{x^5}{y^5}}.$$

$$9. x^{\frac{1}{n}} \div x^{\frac{1}{m}} = x^{\frac{m}{nm}} \div x^{\frac{n}{nm}} = x^{\frac{m-n}{nm}} = \sqrt[nm]{x^{m-n}}.$$

$$10. \sqrt{x^2 - xd - b + d\sqrt{b}} \div \sqrt{x - \sqrt{b}} = x + \sqrt{b} - d.$$

P R O B L E M VIII.

To involve surd quantities to any power.

R U L E.

Multiply the index of the quantity by the index of the power to be raised, and to the result annex the power of the rational parts, and it will be involved as required.

EXAMPLES:

1. The cube of $\sqrt[3]{2} = 2^{\frac{1}{3}} \times 3 = 2^{\frac{3}{3}} = 2^1 = 2$.
2. The square of $\frac{2}{3}a^{\frac{1}{3}} = \frac{4}{9}a^{\frac{1}{3} \times 2} = \frac{4}{9}a^{\frac{2}{3}}$
 $= \frac{4}{9}a^{\frac{2}{3}}$.
3. The 4th power of $\frac{1}{2}\sqrt{\frac{2}{3}} = \frac{1}{16} \times \frac{2}{3}^{\frac{1}{2} \times 4}$
 $= \frac{1}{16} \times \frac{2}{3}^2 = \frac{1}{16} \times \frac{4}{9} = \frac{1}{36}$.
4. The square of $\frac{a}{b} \times \sqrt[3]{\frac{x}{c}} = \frac{a^2}{b^2} \times \sqrt[3]{\frac{x}{c}^2} = \frac{a^2}{b^2} \times \sqrt[3]{\frac{x^2}{c^2}}$
 $= \frac{a^2}{b^2} \times \sqrt[3]{\frac{x^2}{c^2}}$.
5. The n th power of $a^{\frac{1}{m}} = a^{\frac{1}{m} \times n} = a^{\frac{n}{m}} = a^{\frac{1}{m} \times n}$.
6. The n th power of $x + y = x + y = x + y$.
7. The square of $3 + \sqrt{5} = 14 + 6\sqrt{5}$.
8. The cube of $x - \sqrt{y} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$.

PROBLEM IX.

To find the roots of surd quantities.

RULE.

Divide the index of the given quantity by the index of the root to be extracted, and to the result add the root of the rational part, and it will give the root required.

E X A M P L E S :

1. The square root of $3^{\frac{1}{3}} = 3^{\frac{1}{3} \div 2} = 3^{\frac{1}{6}}$.

2. The cube root of $\frac{8}{27} \times a^3 = \frac{2}{3} \times a^{3 \div 3} = \frac{2}{3} a$.

3. The cube root of $\frac{a}{b} \times \sqrt[3]{\frac{x}{y}}^{\frac{1}{2}} = \sqrt[3]{\frac{a}{b}}^{\frac{1}{3}} \times \sqrt[3]{\frac{x}{y}}^{\frac{1}{6}}$
 $= \sqrt[3]{\frac{a}{b}}^{\frac{2}{6}} \times \sqrt[3]{\frac{x}{y}}^{\frac{1}{6}} = \sqrt[3]{\frac{a^2 x}{b^2 y}}^{\frac{1}{6}}$.

4. The 4th root of $3x^3 = x^{\frac{3}{4}} \times 3^{\frac{1}{4}} = 3^{\frac{1}{4}} \times x^{\frac{3}{4}}$.

5. The n th root of $x^{\frac{r}{m}} = x^{\frac{r}{nm}} = \sqrt[nm]{x^r}$.

6. The n th root of $\sqrt[n]{x+y}^m = \sqrt[n]{x+y}^{\frac{nm}{n}} = \sqrt[n]{x+y}^{\frac{1}{n}}$.

7. The square root of $x^2 - 4x\sqrt{a} + 4a = x - 2\sqrt{a}$.

8. The n th root of $x - 2\sqrt{x+y} = x - 2\sqrt{x+y}^{\frac{1}{n}}$.

O F I N F I N I T E S E R I E S .

An infinite series is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a surd quantity; and is such, as, being continued, would run on *ad infinitum*, in the manner of a decimal fraction. And, by obtaining a few of the first terms, the law of the progression will be manifest, so that the series may be continued, without actually performing the whole operation.

P R O B L E M I .

To reduce fractional quantities into infinite series.

R. U. L. E.

Divide the numerator by the denominator, as in common division; and the operation continued, as far as shall be thought necessary, will give the series required.

E X A M P L E S :

1. Let $\frac{ax}{a-x}$ be proposed to be thrown into an infinite series.

$$a-x) ax \dots (x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} \&c.$$

$$\frac{ax - x^2}{a}$$

$$\frac{x^2 - \frac{x^3}{a}}{a}$$

$$\frac{\frac{x^3}{a} - \frac{x^4}{a^2}}{a}$$

$$\frac{\frac{x^4}{a^2} - \frac{x^5}{a^3}}{a}$$

$$\frac{\frac{x^5}{a^3} - \frac{x^6}{a^4}}{a}$$

$$\frac{\frac{x^6}{a^4} - \frac{x^7}{a^5}}{a}$$

$$\frac{x^7}{a^5} \&c.$$

2. Let $\frac{a^2}{a^2+2ax+x^2}$ be proposed to be thrown in-
to an infinite series.

$$a^2+2ax+x^2) a^2 \quad . \quad . \quad . \quad (1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} \&c.$$

$$\underline{a^2+2ax+x^2}$$

$$-2ax-x^2$$

$$\underline{-2ax-4x^2-\frac{2x^3}{a}}$$

$$3x^2+\frac{2x^3}{a}$$

$$3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}$$

$$\underline{-\frac{4x^3}{a}-\frac{3x^4}{a^2}}$$

$$\underline{\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}}$$

$$\underline{\frac{5x^4}{a^2}+\frac{4x^5}{a^3} \&c.}$$

3. To throw $\frac{b}{a+x}$ into an infinite series.

$$Ans. \frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \&c.$$

4. To throw $\frac{a^2}{x+b}$ into an infinite series.

$$Ans. \frac{a^2}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4} + \&c.$$

5. To throw $\frac{1}{1+x^2}$ into an infinite series.

$$Ans. 1 - x^2 + x^4 - x^6 + x^8 - \&c.$$

6. To throw $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x}$ into an infinite series.

$$\text{Ans. } 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}} \text{ \&c.}$$

PROBLEM II.

To reduce a compound surd into an infinite series.

R U L E.

Extract the root as in common arithmetic, and the operation, continued as far as shall be thought necessary, will give the series required.

EXAMPLES:

1. Extract the square root of $a^2 + x^2$ in an infinite series.

$$\frac{a^2 + x^2}{a^2} \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \text{ \&c.} \right)$$

$$\begin{array}{r} 2a + \frac{x^2}{2a} \overline{) x^2} \\ \underline{x^2 + \frac{x^4}{4a^2}} \end{array}$$

$$\begin{array}{r} 2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \overline{) -\frac{x^4}{4a^2}} \\ \underline{-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}} \end{array}$$

$$2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \text{ \&c.} \left. \begin{array}{l} \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\ \frac{x^6}{8a^4} + \frac{x^8}{16a^6} \text{ \&c.} \end{array} \right\}$$

$$\underline{-\frac{5x^8}{64a^6} \text{ \&c.}}$$

2. Throw $\sqrt{x-x^3}^{\frac{1}{2}}$ into an infinite series.

$$\text{Ans. } x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{8} - \frac{x^{\frac{7}{2}}}{16} - \frac{5x^{\frac{9}{2}}}{128} \&c.$$

3. Throw $\sqrt{a^2-x^2}^{\frac{1}{2}}$ into an infinite series.

$$\text{Ans. } a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} \&c.$$

4. Throw $\sqrt{1-x^3}^{\frac{1}{3}}$ into an infinite series.

$$\text{Ans. } 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} \&c.$$

PROBLEM III.

To reduce a binomial surd into an infinite series;
or to extract any root of a binomial.

RULE.

Substitute the particular letters of the binomial, with their proper signs, in the following general form, and it will give the root required; observing that P is the first term, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. the foregoing terms, with their signs.

$$\sqrt[n]{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} (A) + \frac{m}{n} A Q (B) + \frac{m-n}{2n} B Q (C) + \frac{m-2n}{3n} C Q (D) + \frac{m-3n}{4n} D Q (E) \&c.$$

EXAMPLES:

1. To extract the square root of x^2-x^2 , in an infinite series.

F

Here $P=r^2$, $Q=\frac{-x^2}{r^2}$, and $\frac{m}{n}=\frac{1}{2}$.

$$\begin{aligned} \text{Therefore } \sqrt[r^2-x^2]{\frac{1}{2}} &= r + \frac{1}{2} A \times \frac{-x^2}{r^2} - \frac{1}{4} B \times \\ &\frac{-x^2}{r^2} - \frac{3}{6} C \times \frac{-x^2}{r^2} - \frac{5}{8} D \times \frac{-x^2}{r^2} \text{ \&c.} = r - \frac{x^2}{2r^2} A \\ &+ \frac{x^2}{4r^2} B + \frac{3x^2}{6r^2} C + \frac{5x^2}{8r^2} D \text{ \&c.} \end{aligned}$$

And, by restoring the values of $A, B, C, D, \text{ \&c.}$
we shall have $\sqrt[r^2-x^2]{\frac{1}{2}} = r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} -$
 $\frac{5x^8}{128r^7} \text{ \&c.} = \text{series required.}$

2. To extract the cube root of $\frac{a^2}{a^2+x^2}$ in an infinite series.

This expression, reduced, is $\frac{a^{\frac{2}{3}}}{a^2+x^2} = a^{\frac{2}{3}} \times \frac{1}{a^2+x^2}$;
and therefore $P=a^2$, $Q=\frac{x^2}{a^2}$, and $\frac{m}{n}=-\frac{2}{3}$.

$$\begin{aligned} \text{Whence } \sqrt[a^2+x^2]{-\frac{2}{3}} &= a^{-\frac{2}{3}} - \frac{2}{3} A \times \frac{x^2}{a^2} - \frac{5}{6} B \times \\ &\frac{x^2}{a^2} - \frac{8}{9} C \times \frac{x^2}{a^2} - \frac{11}{12} D \times \frac{x^2}{a^2} \text{ \&c.} = \frac{1}{\sqrt[3]{a}} - \frac{2x^2}{3^3\sqrt[3]{a^4}} \\ &+ \frac{5x^4}{9^3\sqrt[3]{a^5}} - \frac{40x^6}{81^3\sqrt[3]{a^7}} + \frac{110x^8}{243^3\sqrt[3]{a^9}} \text{ \&c.} = \frac{1}{\sqrt[3]{a}} \times \left\{ 1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \right. \\ &\left. - \frac{40x^6}{81a^6} + \frac{110x^8}{243a^8} - \right\} \text{ \&c.} \end{aligned}$$

Therefore $\frac{a^{\frac{2}{3}}}{a^2+x^2} = \frac{1}{a^{\frac{1}{3}}} \times \left\{ 1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6} \right.$
 $\left. \text{ \&c. as required.} \right\}$

3. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

$$\text{Ans. } r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3} \text{ \&c.}$$

4. To find the value of $\frac{1}{a^2-x^2}^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } \frac{1}{a} + \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} + \frac{15x^6}{48a^7} \text{ \&c.}$$

5. To find the value of $\frac{a^2}{a+x}^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4} \text{ \&c.}$$

6. To find the value of $\frac{1}{2rx-x^2}^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } \frac{1}{\sqrt{2rx}} \times : 1 + \frac{x}{4r} + \frac{3x^2}{4 \cdot 8r^2} + \frac{3 \cdot 5x^3}{4 \cdot 8 \cdot 12r^3} \text{ \&c.}$$

7. Find the value of $a^2-x^2^{\frac{1}{2}}$, in an infinite series.

$$\text{Ans. } a^{\frac{2}{3}} \times : 1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} \text{ \&c.}$$

8. To find the value of $1-x^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } 1 - \frac{x}{4} - \frac{3x^2}{4 \cdot 8} - \frac{3 \cdot 7x^3}{4 \cdot 8 \cdot 12} - \frac{3 \cdot 7 \cdot 11x^4}{4 \cdot 8 \cdot 12 \cdot 16} \text{ \&c.}$$

9. To find the value of $\frac{a^2+x^2}{a^2-x^2}^{\frac{1}{2}}$ in a series.

$$\text{Ans. } 1 + \frac{x^2}{a^2} - \frac{x^4}{2a^4} + \frac{x^6}{2a^6} \text{ \&c.}$$

10. To find the value of $\frac{ax}{a^2-ax+x^2}$ in a series.

$$\text{Ans. } \frac{x}{a} + \frac{x^3}{a^3} - \frac{x^4}{a^4} - \frac{x^5}{a^5} \text{ \&c.}$$

ARITHMETICAL PROPORTION.

Arithmetical proportion is the relation which two quantities, of the same kind, bear to each other, with respect to their difference.

Four quantities are said to be in *arithmetical proportion*, when the difference between the first and second is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, and $a, a+b, c, c+b$, are *arithmetically proportional*.

Arithmetical progression is when a series of quantities either increase or decrease by the same common difference.

Thus, 2, 4, 6, 8, 10, 12, &c. and $a, a+b, a+2b, a+3b, a+4b, a+5b$, &c. are series in *arithmetical progression*, whose common differences are 2, and b .

The most useful part of arithmetical proportion is contained in the following theorems:

I. If four quantities be in arithmetical proportion, the sum of the two means will be equal to the sum of the two extremes.

Thus, if 2, 5, 7, 10, and a, b, c, d , are in *arithmetical proportion*, then will $2+10=5+7$, and $a+d=b+c$.

II. In any arithmetical series, the last term is equal to the first, more the product of the common difference by the number of terms less one.

Thus the 20th term of 2, 4, 6, 8, 10, 12, &c. is $=2+20-1 \times 2=2+19 \times 2=2+38=40$,

And the n th term of $a, a+x, a+2x, a+3x, a+4x$, &c. is $=a+n-1 \times x=a+n-1x$.

III. The sum of any series of quantities in arithmetical progression, is equal to the sum of the two extremes multiplied by half the number of terms.

Thus the sum of 1, 2, 3, 4, 5, 6, &c. continued to the 20th term, is $=\frac{1+20 \times 20}{2}=\frac{21 \times 20}{2}=21 \times 10=210$.

And the sum of n terms of $a, a+x, a+2x, a+3x,$
 &c. to $a+nx$, is $= \frac{a+a+n \times n}{2} = a + \frac{nn}{2}$.

GEOMETRICAL PROPORTION.

Geometrical proportion is that relation of two quantities, of the same kind, which arises from considering what part the one is of the other, or how often it is contained in it.

In four proportional quantities, the first and third are called the *antecedents*, and the second and fourth the *consequents*.

Ratio is the quotient which arises from dividing the antecedent by the consequent, or the consequent by the antecedent.

Four quantities are said to be *proportional*, when the first is the same part or multiple of the second, as the third is of the fourth.

Thus, 2, 8, 3, 12, and a, ar, b, br , are geometrical proportionals.

Direct proportion is when the same relation subsists between the first term and the second, as between the third and the fourth.

Thus, 3, 6, 5, 10, and x, ax, y, ay , are in direct proportion.

Reciprocal, or inverse proportion, is when one quantity increases in the same proportion that another diminishes.

Thus, 2, 6, 9, 3, and a, ar, br, b , are in inverse proportion.

A series of quantities are said to be in *geometrical progression*, when the first has the same ratio to the second, as the second to the third, and as the third to the fourth, &c.

54 GEOMETRICAL PROPORTION.

Thus, 2, 4, 8, 16, 32, 64, &c. and $a, ar, ar^2, ar^3, ar^4, ar^5, \&c.$ are series in geometrical progression.

The most necessary part of geometrical proportion is contained in the following theorems.

I. If four quantities be in geometrical proportion, the product of the two means will be equal to that of the two extremes.

Thus, if 2, 4, 6, 12, and a, ar, b, br , be geometrically proportional, then will $2 \times 12 = 4 \times 6$, and $a \times br = b \times ar$.

II. If four quantities be in geometrical proportion, the rectangle of the means divided by either of the extremes will give the other extreme.

Thus, if 3, 9, 5, 15, and x, ax, y, ay , are geometrical proportionals, then will $\frac{9 \times 5}{3} = 15$, and $\frac{ax \times y}{ay} = x$.

III. The sum of any series of quantities in geometrical progression, is found by multiplying the last term by the ratio, and dividing the difference of this product and the first term by the ratio less one.

Thus, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, is $\frac{512 \times 2 - 2}{2 - 1} = 1024 - 2 = 1022$.

And the sum of n terms of $a, ar, ar^2, ar^3, ar^4, \&c.$ to ar^{n-1} , is $\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1}$.

IV. If four quantities, a, b, c, d , or 2, 6, 5, 15, are proportional, then will any of the following forms of those quantities be also proportional.

1. directly $a:b::c:d$ or $2:6::5:15$.
2. inversely $b:a::d:c$ or $6:2::15:5$.
3. alternately $a:c::b:d$ or $2:5::6:15$.
4. compoundedly $a:a+b::c:c+d$ or $2:8::5:20$.
5. dividedly $a:b-a::c:d-e$ or $2:4::5:10$.
6. mixtly $b+a:b-a::d+c:d-c$ or $8:4::20:10$.
7. by multiplication $ra:rb::c:d$ or $2 \times 3(6):6 \times 3(18)::5:15$.

8. by division $\frac{a}{r} : \frac{b}{r} :: c : d$ or $\frac{2}{2}(1) : \frac{6}{2}(3) ::$

5 : 15 :

SIMPLE EQUATIONS.

An equation is, when two equal quantities, differently expressed, are compared together, by means of the sign = placed between them.

Thus $12 - 5 = 7$ is an equation, expressing the equality of the quantities $12 - 5$ and 7 .

A simple equation is that which contains only one unknown quantity, without including its power.

Thus $x - a + b = c$ is a simple equation, containing only the unknown quantity x .

Reduction of equations is the method of finding the value of the unknown quantity; which is shewn in the following rules.

RULE I.

Any quantity may be transposed from one side of the equation to the other, by changing its sign.

Thus, if $x + 3 = 7$, then will $x = 7 - 3 = 4$.

And, if $x - 4 + 6 = 8$, then will $x = 8 + 4 - 6 = 6$.

Also, if $x - a + b = c - d$, then will $x = c - d + a - b$.

And, in like manner, if $4x - 8 = 3x + 20$, then will $4x - 3x = 20 + 8$, or $x = 28$.

RULE II.

If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms of the equation by it.

56 SIMPLE EQUATIONS.

Thus, if $ax=ab-a$, then will $x=b-1$.

And, if $2x+4=16$, then will $x+2=8$, and $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, then will $x+2b=\frac{3c^2}{a}$, and $x=\frac{3c^2}{a}-2b$.

R U L E. III.

If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, then will $x=10+6=16$.

And, if $\frac{x}{a}=b+c-d$, then will $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, then will $2x-6=18+12$, and $2x=18+12+6=36$, or $x=\frac{36}{2}=18$.

R U L E. IV.

The unknown quantity in any equation may be made free from surds, by transposing the rest of the terms according to the rule, and then involving each side to such a power as is denoted by the index of the said surd.

Thus, if $\sqrt{x}-2=6$, then will $\sqrt{x}=6+2=8$, and $x=8^2=64$.

And, if $\sqrt{4x+16}=12$, then will $4x+16=144$, and $4x=144-16=128$, or $x=\frac{128}{4}=32$.

SIMPLE EQUATIONS. 57

In like manner, if $\sqrt[3]{2x+3}+4=8$, then will
 $\sqrt[3]{2x+3}=8-4=4$, and $2x+3=4^3=64$, and
 $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

R U L E V.

If that side of the equation which contains the unknown quantity be a complete power, it may be reduced by extracting the root of the said power from both sides of the equation.

Thus, if $x^2+6x+9=25$, then will $x+3=\sqrt{25}=5$,
 or $x=5-3=2$.

And, if $3x^2-9=21+3$, then will $3x^2=21+3$
 $+9=33$, and $x^2=\frac{33}{3}=11$, or $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, then will $2x^2+$
 $30=60$, and $x^2+15=30$, or $x^2=30-15=15$, or
 $x=\sqrt{15}$.

R U L E VI.

Any analogy, or proportion, may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x:16::5:10$, then will $3x\times 10=16\times 5$,
 and $30x=80$, or $x=\frac{80}{30}=2\frac{2}{3}$.

And, if $\frac{2x}{3}:a::b:c$, then will $\frac{2cx}{3}=ab$,
 and $2cx=3ab$, or $x=\frac{3ab}{2c}$.

58 SIMPLE EQUATIONS.

In like manner, if $12-x : \frac{x}{2} :: 4 : 1$, then will

$$12-x = \frac{4x}{2} = 2x, \text{ and } 2x+x=12, \text{ or } x = \frac{12}{3} = 4.$$

R U L E VII.

If any quantity be found on both sides of the equation with the same sign, it may be taken away from them both; and if every term in an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $4x+a=b+a$, then will $4x=b$, and $x = \frac{b}{4}$.

And, if $3ax+5ab=8ac$, then will $3x+5b=8c$, and $x = \frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3} - \frac{8}{3} = \frac{16}{3} - \frac{8}{3}$, then will $2x=16$, and $x=8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$ to find the value of x .

First $5x-2x=6+15$

then $3x=21$

$$\text{and } x = \frac{21}{3} = 7$$

2. Given $40-6x-16=120-14x$ to find x .

First $14x-6x=120-40+16$

then $8x=96$

$$\text{and, therefore } x = \frac{96}{8} = 12.$$

SIMPLE EQUATIONS. 59

3. Let $5ax - 3b = 2dx + c$ be given, to find x .

First, $5ax - 2dx = c + 3b$

or $5a - 2d \times x = c + 3b$

and therefore $x = \frac{c + 3b}{5a - 2d}$.

4. Let $3x^2 - 10x = 8x + x^2$ be given to find x .

First, $3x - 10 = 8 + x$

and then $3x - x = 8 + 10$

therefore $2x = 18$, and $x = \frac{18}{2} = 9$.

5. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$, to find x .

First, dividing the whole by $3ax^2$, we shall have

$$2x - 4b = x + 2$$

and then $2x - x = 2 + 4b$

whence $x = 2 + 4b$.

6. Let $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$, be given to find x ,

First, $x - \frac{2x}{3} + \frac{2x}{4} = 20$

and then $3x - 2x + \frac{6x}{4} = 60$

and $12x - 8x + 6x = 240$

therefore $10x = 240$

and $x = \frac{240}{10} = 24$.

7. Given $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x+19}{2}$ to find x .

First, $x - 3 + \frac{2x}{3} = 40 - x - 19$

and then $3x - 9 + 2x = 120 - 3x - 57$

and therefore $3x + 2x + 3x = 120 - 57 + 9$

that is $8x = 72$, or $x = \frac{72}{8} = 9$.

60 SIMPLE EQUATIONS.

8. Let $\sqrt{\frac{2}{3}x+5}=7$, be given to find x .

First, $\sqrt{\frac{2}{3}x+5}=7-5=2$

and then $\frac{2}{3}x+5=2^2=4$

and $2x=12$, or $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given to find x .

First, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

and then $x\sqrt{a^2+x^2}=a^2-x^2$

and $x^2 \times a^2 + x^2 = a^2 - x^2$ $\left[\right]^2 = a^4 - 2a^2x^2 + x^4$

or $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$

whence $a^2x^2 + 2a^2x^2 = a^4$, or $3a^2x^2 = a^4$

and consequently $x^2 = \frac{a^4}{3a^2}$, and $x = \sqrt{\frac{a^4}{3a^2}} = a\sqrt{\frac{1}{3}}$

EXAMPLES FOR PRACTICE.

1. Given $x+18=3x-5$ to find x . *Ans.* $x=11\frac{1}{2}$.

2. Given $3y-a+b=cd$ to find y .

Ans. $y=\frac{cd+a-b}{3}$

3. Given $6-2x+10=20+3x-2$ to find x .

Ans. $x=2$.

4. Given $3ax+\frac{a}{2}-3=bx-a$ to find x .

Ans. $x=\frac{6-3a}{7a-2b}$.

SIMPLE EQUATIONS. 61

5. Given $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d$ to find x .

$$\text{Ans. } x = \frac{abcd}{bc + ac + ab}.$$

6. Given $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{1}{2}$ to find x .

$$\text{Ans. } x = \frac{6}{7}.$$

7. Given $\sqrt{12+x} = 2 + \sqrt{x}$ to find x .

$$\text{Ans. } x = 4.$$

8. Given $\sqrt{a^2+x^2} = b^2 + x^2$ to find x .

$$\text{Ans. } x = \sqrt{\frac{b^4 - a^4}{2a^2}}.$$

9. Given $x + a = \sqrt{a^2 + x} \sqrt{b^2 + x^2}$ to find x .

$$\text{Ans. } x = \frac{b^2}{4a} - a.$$

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two simple equations containing them to a single one.

RULE I.

1. Observe which of the unknown quantities is the least-involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will arise a new equation with only one unknown quantity in it, whose value may be found as before.

62 SIMPLE EQUATIONS.

EXAMPLES:

1. Given $\begin{cases} 2x+3y=23 \\ 5x-2y=10 \end{cases}$ to find x and y .

From the first equation $x = \frac{23-3y}{2}$,

and from the second $x = \frac{10+2y}{5}$,

and consequently $\frac{23-3y}{2} = \frac{10+2y}{5}$,

or $115-15y=20+4y$,

or $19y=115-20=95$,

and $y = \frac{95}{19} = 5$,

whence $x = \frac{23-15}{2} = 4$.

2. Given $\begin{cases} x+y=a \\ x-y=b \end{cases}$ to find x and y .

From the first equation $x=a-y$,

and from the second $x=b+y$,

Therefore $a-y=b+y$, or $2y=a-b$,

and consequently $y = \frac{a-b}{2}$,

and $x = a-y = a - \frac{a-b}{2} = \frac{a+b}{2}$.

3. Given $\begin{cases} \frac{x}{2} + \frac{y}{3} = 7 \\ \frac{x}{3} + \frac{y}{2} = 8 \end{cases}$ to find x and y .

From the first equation $x = 14 - \frac{2y}{3}$,

and from the second $x = 24 - \frac{3y}{2}$,

Therefore $14 - \frac{2y}{3} = 24 - \frac{3y}{2},$

and $42 - 2y = 72 - \frac{9y}{2},$

or $84 - 4y = 144 - 9y;$

whence $5y = 144 - 84 = 60,$

and $y = \frac{60}{5} = 12,$

and $x = 14 - \frac{2y}{3} = 14 - \frac{24}{3} = 6.$

4. Given $4x + y = 34,$ and $4y + x = 16,$ to find x and $y.$ *Ans.* $x = 8,$ and $y = 2.$

5. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20},$ and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120},$ to find x and $y.$ *Ans.* $x = \frac{1}{2}$ and $y = \frac{1}{3}.$

6. Given $x + y = s,$ and $x^2 - y^2 = d,$ to find x and $y.$

Ans. $x = \frac{s^2 + d}{2s},$ and $y = \frac{s^2 - d}{2s}.$

R U L E II.

1. Consider which of the unknown quantities you would first exterminate, and let its value be found in that equation where it is the least involved.

2. Substitute the value, thus found, for its equal in the other equation, and there will arise a new equation, with only one unknown quantity, whose value may be found as before.

64 SIMPLE EQUATIONS.

EXAMPLES:

1. Given $\begin{cases} x+2y=17 \\ 3x-y=2 \end{cases}$ to find x and y .

From the first equation $x=17-2y$,
And this value substituted for x in the second, gives

$$17-2y \times 3 - y = 2, \\ \text{or } 51-6y-y=2, \text{ or } 51-7y=2; \\ \text{that is } 7y=51-2=49;$$

$$\text{whence } y=\frac{49}{7}=7, \text{ and } x=17-2y=17-14=3.$$

2. Given $\begin{cases} x+y=13 \\ x-y=3 \end{cases}$ to find x and y .

From the first equation $x=13-y$,
And this value being substituted for x in the 2d,
gives $13-y-y=3$, or $13-2y=3$,
that is $2y=13-3=10$,

$$\text{or } y=\frac{10}{2}=5, \text{ and } x=13-y=13-5=8.$$

3. Given $\begin{cases} a:b::x:y \\ x^2+y^2=c \end{cases}$ to find x and y .

The first analogy turned into an equation
is $bx=ay$, or $x=\frac{ay}{b}$,

And this value of x substituted in the 2d,

$$\text{gives } \left(\frac{ay}{b}\right)^2 + y^2 = c, \text{ or } \frac{a^2y^2}{b^2} + y^2 = c,$$

$$\text{or } a^2y^2 + b^2y^2 = cb^2, \text{ or } y^2 = \frac{cb^2}{a^2 + b^2},$$

$$\text{and therefore } y = \sqrt{\frac{cb^2}{a^2 + b^2}}^{\frac{1}{2}}, \text{ and } x = \sqrt{\frac{ca^2}{a^2 + b^2}}^{\frac{1}{2}}.$$

4. Given $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$, to find x and y .
Ans. $x=7$ and $y=14$.

5. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$, and $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y-x}{4} + 27$, to find x and y .
Ans. $x=60$ and $y=40$.

6. Given $a:b::x:y$, and $x^3 - y^3 = d$, to find x and y .

$$\text{Ans. } \sqrt[3]{\frac{da^3}{a^3 - b^3}} = x, \text{ and } \sqrt[3]{\frac{db^3}{a^3 - b^3}} = y.$$

R U L E III.

Let the given equations be multiplied or divided by such numbers or quantities as will make the term which contains one of the unknown quantities to be the same in both equations; and then by adding or subtracting the equations, according as is required, there will arise a new equation with only one unknown quantity, as before.

E X A M P L E S:

1. Given $\begin{cases} 3x + 5y = 40 \\ x + 2y = 14 \end{cases}$ to find x and y .

First, multiply the 2d equation by 3,

and we shall have $3x + 6y = 42$,

Then, from this last equation subtract the first,

and it will give $6y - 5y = 42 - 40$, or $y = 2$,

and therefore $x = 14 - 2y = 14 - 4 = 10$.

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2. Given $\begin{cases} 5x-3y=9 \\ 2x+5y=16 \end{cases}$ to find x and y .

Let the 1st equation be multiplied by 2, and the 2d by 5, and we shall have

$$10x-6y=18$$

$$10x+25y=80$$

and if the former of these be subtracted from the latter,

$$\text{it will give } 31y=62, \text{ or } y=\frac{62}{31}=2,$$

$$\text{and consequently } x=\frac{9+3y}{5}, \text{ by the first equation,}$$

$$\text{or } x=\frac{9+6}{5}=\frac{15}{5}=3.$$

Another method.

Multiply the first equation by 5, and the second by 3, and we shall have

$$25x-15y=45$$

$$6x+15y=48$$

Now, let these two equations be added together,

$$\text{and it will give } 31x=93, \text{ or } x=\frac{93}{31}=3,$$

$$\text{and consequently } y=\frac{16-2x}{5}, \text{ by the 2d equation,}$$

$$\text{or } y=\frac{16-6}{5}=\frac{10}{5}=2 \text{ as before.}$$

MISCELLANEOUS EXAMPLES:

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$,
to find x and y . *Ans.* $x=19$ and $y=3$.

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$
to find x and y . *Ans.* $x=5$ and $y=2$.

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3. Given $\frac{2x+3y}{6} + \frac{x}{3} = 8$, and $\frac{7y-3x}{2} - y = 11$,
to find x and y . *Ans.* $x=6$ and $y=8$.

4. Given $ax+by=c$, and $dx+ey=f$, to find x
and y . *Ans.* $x = \frac{ce-bf}{ae-db}$ and $y = \frac{af-dc}{ae-db}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three simple equations, containing them, to a single one.

RULE.

1. Let x , y , and z , be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the first value of x with the second, and an equation will arise involving only y and z .

4. In like manner, compare the first value of x with the third, and another equation will arise involving only y and z .

5. Find the values of y and z from these two equations, according to the former rules, and x , y , and z will be exterminated as required.

Note, Any number of unknown quantities may be exterminated in nearly the same manner, but there are often much shorter methods for performing the operation, which will be best learnt from practice.

68 SIMPLE EQUATIONS.

EXAMPLES:

1. Given $\left\{ \begin{array}{l} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10 \end{array} \right\}$ to find $x, y,$ and z .

From the first equation $x = 29 - y - z$.

From the second $x = 62 - 2y - 3z$.

From the third $x = 20 - \frac{2y}{3} - \frac{z}{2}$.

whence $29 - y - z = 62 - 2y - 3z$,

and $29 - y - z = 20 - \frac{2y}{3} - \frac{z}{2}$,

but, from the first of these equations, $y = 62 - 29 - 2z$
 $= 33 - 2z$,

and from the second $y = 27 - \frac{3z}{2}$,

therefore $33 - 2z = 27 - \frac{3z}{2}$, or $z = 12$,

and $y = 62 - 29 - 2z = 62 - 29 - 24 = 9$,

and $x = 29 - y - z = 29 - 12 - 9 = 8$.

2. Given $\left\{ \begin{array}{l} \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62 \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47 \\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38 \end{array} \right\}$ to find $x, y,$ and z .

First, the given equations, cleared of fractions, become

$$12x + 8y + 6z = 1488$$

$$20x + 15y + 12z = 2820$$

$$30x + 24y + 20z = 4560$$

Then, if the second of these equations be subtracted from double the first, and three times the third from five times the second, we shall have

$$\begin{aligned} 4x + y &= 156 \\ 10x + 3y &= 420 \end{aligned}$$

And again, if the second of these be subtracted from three times the first, it will give

$$12x - 10x = 468 - 420, \text{ or } x = \frac{48}{2} = 24$$

Therefore $y = 156 - 4x = 60$, and $z = \frac{1488 - 8y - 12x}{6}$
 $= 120$.

3. Given $x + y + z = 31$, $x + y - z = 25$, and $x - y - z = 9$, to find x , y , and z .

Ans. $x = 20$, $y = 8$, and $z = 3$.

4. Given $x + y = a$, $x + z = b$, and $y + z = c$, to find x , y , and z .

5. Given $\begin{cases} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{cases}$ to find x , y , and z .

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, such that their sum shall be 40, and their difference 16.

*Let x denote the least of the two numbers required,
 then will $x + 16 =$ to the greater,
 and $x + x + 16 = 40$ by the question,
 that is $2x + 16 = 40$,
 that is $2x = 40 - 16 = 24$,*

70 SIMPLE EQUATIONS.

or $x = \frac{24}{2} = 12 = \text{least number,}$

and $x + 16 = 12 + 16 = 28 = \text{greater number required.}$

2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x = \text{number required,}$

then will its $\frac{1}{3}$ part be $\frac{x}{3}$, and its $\frac{1}{4}$ part $\frac{x}{4}$;

and therefore $\frac{x}{3} - \frac{x}{4} = 16$ by the question,

that is $x - \frac{3x}{4} = 48$, or $4x - 3x = 192$;

whence $x = 192$ the number required.

3. Divide 1000l. between A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x = B$'s share of the given sum,

then will $x + 72 = A$'s share,

and $x + 172 = C$'s share,

And the sum of all their shares $x + x + 72 + x + 172$,

or $3x + 244 = 1000$ by the question,

that is $3x = 1000 - 244 = 756$,

or $x = \frac{756}{3} = 252\text{l.} = B$'s share,

and $x + 72 = 252 + 72 = 324\text{l.} = A$'s share,

and $x + 172 = 252 + 172 = 424\text{l.} = C$'s share.

252l.

324l.

424l.

1000l. the proof.

5. A prize of 1000l. is to be divided between two persons, whose shares therein are in the proportion of 7 to 9: required the share of each.

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Let x = first person's share,
 then will $1000 - x$ = second person's share,
 and $x : 1000 - x :: 7 : 9$, by the question,
 that is $9x = 1000 - x \times 7 = 7000 - 7x$,
 or $16x = 7000$,

whence $x = \frac{7000}{16} = 437\text{l. } 10\text{s.} = 1\text{st share}$,

and $1000 - x = 1000 - 437\text{l. } 10\text{s.} = 562\text{l. } 10\text{s. } 2\text{d share}$.

6. The paving of a square at 2s. a yard, cost as much as the inclosing it at 5s. a yard : required the side of the square.

Let x = side of the square sought,
 then $4x$ = yards of inclosure,
 and x^2 = yards of pavement ;
 whence $4x \times 5 = 20x$ = price of inclosing,
 and $x^2 \times 2 = 2x^2$ = price of paving.

But $2x^2 = 20x$ by the question,
 therefore $x^2 = 10x$, and $x = 10$ = length of the side required.

7. A labourer engaged to serve for 40 days upon these conditions, that for every day he worked he should receive 20 d. but for every day that he played, or was absent, he was to forfeit 8 d. now at the end of the time he had to receive 1 l. 11 s. 8 d. the question is, to find how many days he worked, and how many he was idle.

Let x be the number of days he worked,
 then will $40 - x$ be the number of days he was idle,
 also $x \times 20 = 20x$ = sum earned,
 and $40 - x \times 8 = 320 - 8x$ = sum forfeited,
 whence $20x - 320 - 8x = 380\text{d.} (= 1\text{l. } 11\text{s. } 8\text{d.})$ by
 the question, that is $20x - 320 + 8x = 380$
 or $28x = 380 + 320 = 700$,

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and $x = \frac{700}{28} = 25 =$ number of days he worked,

and $40 - x = 40 - 25 = 15 =$ number of days he was idle.

8. Out of a cask of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full: How much did it hold?

Let it be supposed to have held x gallons,

Then it would have leaked $\frac{x}{3}$ gallons,

and consequently there had been taken away $21 + \frac{x}{3}$ gallons.

But $21 + \frac{x}{3} = \frac{x}{2}$ by the question,

That is $63 + x = \frac{3x}{2}$,

or $126 + 2x = 3x$

hence $3x - 2x = 126$

or $x = 126 =$ number of gallons required.

9. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be represented by $\frac{x}{y}$,

Then will $\frac{x+1}{y} = \frac{1}{3}$,

And $\frac{x}{y+1} = \frac{1}{4}$,

or $3x + 3 = y$,

and $4x = y + 1$,

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hence $4x - 3x - 3 = y + 1 - y$

that is $x - 3 = 1$

or $x = 4$, and $y = 3x + 3$

$= 12 + 3 = 15$

So that $\frac{4}{15} = \text{fraction required.}$

10. A market-woman bought in a certain number of eggs, at 2 a penny, and as many at three a penny, and sold them all out again at the rate of 5 for 2 pence, and, by so doing, lost 4 d. what number of eggs had she?

Let $x = \text{number of eggs of each sort.}$

Then will $\frac{x}{2} = \text{price of the first sort,}$

and $\frac{x}{3} = \text{price of the second sort.}$

But $5 : 2 :: 2x \text{ (the whole number of eggs) : } \frac{4x}{5};$

therefore $\frac{4x}{5} \text{ price of both sorts together, at five for 2d.}$

And $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4 \text{ by the question;}$

that is $x + \frac{2x}{3} - \frac{8x}{5} = 8;$

or $3x + 2x - \frac{24x}{5} = 24;$

or $15x + 10x - 24x = 120$

Whence $x = 120 = \text{number of eggs of each sort required.}$

11. If A can do a piece of work alone in ten days, and B in thirteen; set them both about it together, in what time will it be finished?

Let the time sought be denoted by x ,

Then 10 days: 1 work :: x days: $\frac{x}{10}$,

and 13 days: 1 work :: x days: $\frac{x}{13}$,

hence $\frac{x}{10}$ = part done by A in x days;

and $\frac{x}{13}$ = part done by B in x days.

Consequently $\frac{x}{10} + \frac{x}{13} = 1$;

That is $\frac{13x}{10} + x = 13$, or $13x + 10x = 130$;

And therefore $23x = 130$, or $x = \frac{130}{23} = 5 \frac{15}{23}$ days, the time required.

12. If one agent A, alone, can produce an effect e , in the time a , and another agent B, alone, in the time b ; in what time will they both together produce the same effect?

Let the time sought be denoted by x .

Then $a : e :: x : \frac{ex}{a}$ = part of the effect produced by A,

and $b : e :: x : \frac{ex}{b}$ = part of the effect produced by B,

whence $\frac{ex}{a} + \frac{ex}{b} = e$ by the question;

or $\frac{x}{a} + \frac{x}{b} = 1$;

that is $x + \frac{ax}{b} = a$;

or $bx + ax = ba$;

and consequently $x = \frac{ba}{b+a}$ = time required.

QUESTIONS FOR PRACTICE.

1. What two numbers are those whose difference is 7, and sum 33. *Ans. 13 and 20.*

2. To divide the number 75 into two such parts, that three times the greater may exceed seven times the lesser by 15. *Ans. 54 and 21.*

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole plus 25 gallons was wine, and $\frac{1}{3}$ part minus 5 gallons was cyder: how many gallons were there of each?

Ans. 85 of wine, and 35 of cyder.

4. A bill of 120 l. was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; how many were there of each?

Ans. 50 of each.

5. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Ans. in 10 days.*

6. At a certain election 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each?

Ans. 233 for one, and 142 for the other.

7. There is a fish whose tail weighs 9 lb. his head weighs much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish?

Ans. 72 lb.

8. What number is that from which if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40. *Ans. 65.*

9. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length?

Ans. 24 feet.

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I found 66 guineas left in my bag; what was in it at first? *Ans. 120 guineas.*

11. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each? *Ans. A's=84, B's=42, and C's=14.*

12. Two persons, A and B, lay out equal sums of money in trade; A gains 126 l. and B loses 87 l. and A's money is now double of B's; what did each lay out? *Ans. 300 l.*

13. A person bought a chaise, horse and harness, for 60 l. the horse came to twice the price of the harness, and the chaise to twice the price of the horse and the harness; what did he give for each? *Ans. 13 l. 6 s. 8 d. for the horse, 6 l. 13 s. 4 d. for the harness, and 40 l. for the chaise.*

14. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly, but B, by spending 50 l. *per annum* more than A, at the end of 4 years finds himself 100 l. in debt; what is their income? *Ans. 125 l.*

15. A gentleman has two horses, and a saddle worth 50 l. now if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse? *Ans. One 30 l. and the other 40 l.*

16. To divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other? *Ans. The parts are 8, 12, and 16.*

17. A footman agreed to serve his master for 8 l. a year and a livery, but was turned away at the end of 7 months, and received only 2 l. 13 s. 4 d. and his livery; what was its value? *Ans. 4 l. 16 s.*

18. A gentleman was desirous of giving 3d. a-piece to some poor beggars, but found that he had not money enough in his pocket by 8d. he therefore gave them each 2d. and had then 3d. remaining; required the number of beggars?

Ans. 11.

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's three; but 2 of the greyhound's leaps are as much as three of the hare's; how many leaps must the greyhound take to catch the hare?

Ans. 300.

20. A person in play, lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which, he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had, and this done, found he had but 12s. remaining; what had he at first.

Ans. 20s.

21. To divide the number 90 into 4 such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2; the sum, difference, product, and quotient, shall be all equal to each other.

Ans. The parts are 18, 22, 10, and 40 respectively.

22. The hour and minute hand of a clock are exactly together at 12 o'clock, when are they next together?

Ans. 1 ho. 5 $\frac{1}{11}$ min.

23. There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a day, B 8, and C 10; when will they all come together again?

Ans. 73 days.

24. If A can do a piece of work alone in 10 days, and A and B together in 7 days; in what time can B do it alone?

Ans. 23 $\frac{1}{3}$ days.

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25. If three agents, A, B, and C, can produce the effects a, b, c , in the times e, f, g , respectively: in what time would they jointly produce the effect d ?

$$\text{Ans. } \frac{e+f+g \times d}{a+b+c} \text{ time.}$$

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: How many days will it take each person to perform the same work alone?

$$\text{Ans. } A \ 14\frac{34}{49} \text{ days, } B \ 17\frac{23}{41}, \text{ and } C \ 23\frac{7}{31}.$$

QUADRATIC EQUATIONS.

A *simple quadratic equation* is that which involves the square of the unknown quantity only.

An *affected quadratic equation* is that which involves the square of the unknown quantity, together with the product that arises from multiplying it by some known quantity.

Thus $ax^2 = b$, is a *simple quadratic equation*,

And $ax^2 + bx = c$, is an *affected quadratic equation*.

The rule for a simple quadratic equation has been given already.

All affected quadratic equations fall under the three following forms.

$$1. \ x^2 + ax = b$$

$$2. \ x^2 - ax = b$$

$$3. \ x^2 - ax = -b.$$

The rule for finding the value of x , in each of these equations, is as follows:

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R U L E . *

1. Transpose all the terms that involve the unknown quantity to one side of the equation, and the known terms to the other side, and let them be ranged according to their dimensions.
2. When the square of the unknown quantity has any co-efficient prefixed to it, let all the rest of the terms be divided by that co-efficient.
3. Add the square of half the co-efficient of the second term to both sides of the equation, and that side which involves the unknown quantity will then be a complete square.
4. Extract the square root from both sides of the equation, and the value of the unknown quantity will be determined, as required.

* The square root of any quantity may be either + or —, and therefore all quadratic equations admit of two solutions. Thus the square root of $+n^2$ is $+n$ or $-n$; for $+n \times +n$ or $-n \times -n$ are each equal to $+n^2$. So, in the first form, where $x + \frac{a}{2}$ is found $= \sqrt{b + \frac{a^2}{4}}$ the root may be either $+\sqrt{b + \frac{a^2}{4}}$ or $-\sqrt{b + \frac{a^2}{4}}$, since either of them being multiplied by itself will produce $\sqrt{b + \frac{a^2}{4}}$. And this ambiguity is expressed by writing the uncertain sign $\pm$ before $\sqrt{b + \frac{a^2}{4}}$; thus $x = \pm \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$.

80 QUADRATIC EQUATIONS.

Note, 1. The square root of one side of the equation is always equal to the unknown quantity, with half the co-efficient of the second term subjoined to it.

2. All equations wherein there are two terms involving the unknown quantity, and the index of the one is just double that of the other, are solved like quadratics, by completing the square.

Thus, $x^4 + ax^2 = b$, or $x^2 + ax^{\frac{n}{2}} = b$, are the same as quadratics, and the value of the unknown quantity may be determined accordingly.

In the first form, where $x = \pm \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$, the first value of x , viz. $x = + \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ is always affirmative; for since $\frac{a^2}{4} + b$ is greater than $\frac{a^2}{4}$, the greatest square must necessarily have the greatest square root; $\sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ will, therefore, always be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and consequently $+ \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ will always be affirmative.

The second value, viz. $x = - \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ will always be negative, because it is composed of two negative terms. Therefore when $x^2 + ax = b$, we shall have $x = + \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ for the affirmative value of x , and $x = - \sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$ for the negative value of x .

QUADRATIC EQUATIONS. 81

EXAMPLES:

1. Given $x^2 + 4x = 140$ to find x .

First, $x^2 + 4x + 4 = 140 + 4 = 144$, by completing the square;

then $\sqrt{x^2 + 4x + 4} = \sqrt{144}$ by extracting the root;

or $x + 2 = 12$,

and therefore $x = 12 - 2 = 10$.

2. Given $x^2 - 6x + 8 = 80$, to find x .

First, $x^2 - 6x = 80 - 8 = 72$ by transposition;

Then $x^2 - 6x + 9 = 72 + 9 = 81$ by completing the square;

And $x - 3 = \sqrt{81} = 9$ by extracting the root;

Therefore $x = 9 + 3 = 12$.

In the second form, where $x = \pm \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ the

first value, viz. $x = + \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ is always affirmative, since it is composed of two affirmative terms. The

second value, viz. $x = - \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ will always be

negative; for since $b + \frac{a^2}{4}$ is greater than $\frac{a^2}{4}$, the square root

of $b + \frac{a^2}{4}$ ($\sqrt{b + \frac{a^2}{4}}$) will be greater than $\sqrt{\frac{a^2}{4}}$, or its

equal $\frac{a}{2}$; and consequently $- \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ is always

a negative quantity. Therefore when $x^2 - ax = b$, we shall

have $x = + \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ for the affirmative value of x , and

$x = - \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}$ for the negative value of x .

82 QUADRATIC EQUATIONS.

3. Given $2x^2 + 8x - 20 = 70$, to find x .

First, $2x^2 + 8x = 70 + 20 = 90$ by transposition.

Then $x^2 + 4x = 45$ by dividing by 2,

And $x^2 + 4x + 4 = 49$ by completing the square;

Whence $x + 2 = \sqrt{49} = 7$ by extracting the root.

And consequently $x = 7 - 2 = 5$.

4. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$, to find x .

Here, $x^2 - x + 2 = 1\frac{7}{9}$ by dividing by 3,

And $x^2 - x = 1\frac{7}{9} - 2$ by transposition;

Also $x^2 - x + \frac{1}{4} = 1\frac{7}{9} - 2 + \frac{1}{4} = \frac{1}{36}$ by completing the squ.

And $x - \frac{1}{2} = \sqrt{\frac{1}{36}} = \frac{1}{6}$ by evolution;

Therefore $x = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$.

In the third form, where $x = \sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$, both the values of x will be positive, supposing $\frac{a^2}{4}$ is greater than b .

For the first value, viz. $x = +\sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$ is evidently affirmative, being composed of two affirmative terms. The

second value, viz. $x = -\sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$ is also affirmative;

for since $\frac{a^2}{4}$ is greater than $\frac{a^2}{4} - b$, therefore $\sqrt{\frac{a^2}{4}}$ or $\frac{a}{2}$

is greater than $\sqrt{\frac{a^2}{4} - b}$, and consequently $-\sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$

will always be an affirmative quantity. Therefore, when

$x^2 - ax = -b$, we shall have $x = +\sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$ for

the affirmative value of x , and $-\sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$ for the negative value of x .

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5. Given $\frac{x^2}{2} - \frac{x}{3} + 20\frac{1}{2} = 42\frac{2}{3}$, to find x .

Here $\frac{x^2}{2} - \frac{x}{3} = 42\frac{2}{3} - 20\frac{1}{2} = 22\frac{1}{6}$ by transposition,

And $x^2 - \frac{2x}{3} = 44\frac{1}{3}$ by dividing by $\frac{1}{2}$;

Whence $x^2 - \frac{2x}{3} + \frac{1}{9} = 44\frac{1}{3} + \frac{1}{9} = 44\frac{4}{9}$ by completing the square,

And $x - \frac{1}{3} = \sqrt{44\frac{4}{9}} = 6\frac{2}{3}$,

Therefore $x = 6\frac{2}{3} + \frac{1}{3} = 7$.

6. Given $ax^2 + bx = c$, to find x .

First, $x^2 + \frac{b}{a}x = \frac{c}{a}$ by division;

Then $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$ by completing the square;

But in this third form, if b be greater than $\frac{a^2}{4}$ the solution of the proposed question will be impossible. For since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But if b be greater than $\frac{a^2}{4}$, then $\frac{a^2}{4} - b$ is a negative quantity; and consequently $\sqrt{\frac{a^2}{4} - b}$ is impossible, or only imaginary, when $\frac{a^2}{4}$ is less than b ; and therefore in that case $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$ is also impossible or imaginary.

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And $x + \frac{b}{2a} = \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}} = \sqrt{\frac{4ac + b^2}{4a^2}}$ by evolution;

Therefore $x = \pm \sqrt{\frac{4ac + b^2}{4a^2}} - \frac{b}{2a}$.

7. Given $ax^2 - bx + c = d$, to find x .

Here, $ax^2 - bx = d - c$ by transposition,

And $x^2 - \frac{b}{a}x = \frac{d-c}{a}$ by division.

Also $x^2 - \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{d-c}{a} + \frac{b^2}{4a^2}$ by completing the square;

And $x - \frac{b}{2a} = \pm \sqrt{\frac{d-c}{a} + \frac{b^2}{4a^2}}$ by evolution;

Therefore $x = \frac{b}{2a} \pm \sqrt{\frac{d-c}{a} + \frac{b^2}{4a^2}}$.

8. Given $x^4 + 2ax^2 = b$, to find x .

Here, $x^4 + 2ax^2 + a^2 = b + a^2$ by completing the square,

And $x^2 + a = \sqrt{b + a^2}$ by evolution;

Whence $x^2 = \sqrt{b + a^2} - a$,

And consequently $x = \sqrt{\sqrt{b + a^2} - a}$.

9. Given $ax^n - bx^{\frac{n}{2}} - c = -d$, to find x .

First, $ax^n - bx^{\frac{n}{2}} = c - d$ by transposition,

And $x^n - \frac{b}{a}x^{\frac{n}{2}} = \frac{c-d}{a}$ by division,

Also $x^n - \frac{b}{a}x^{\frac{n}{2}} + \frac{b^2}{4a^2} = \frac{c-d}{a} + \frac{b^2}{4a^2}$ by completing the square,

And $x^{\frac{n}{2}} - \frac{b}{2a} = \sqrt{\frac{c-d}{a} + \frac{b^2}{4a^2}}$ by evolution;

QUADRATIC EQUATIONS. 85

Therefore $x^{\frac{n}{2}} = \frac{b}{2a} \pm \sqrt{\frac{c-d}{a} + \frac{b^2}{4a^2}},$

And consequently $x = \frac{b}{2a} \pm \sqrt{\frac{c-d}{a} + \frac{b^2}{4a^2}}^{\frac{2}{n}}$

EXAMPLES FOR PRACTICE.

1. Given $x^2 - 8x + 10 = 19$ to find x . *Ans.* $x = 9$.
2. Given $x^2 - x - 40 = 170$ to find x . *Ans.* $x = 15$.
3. Given $3x^2 + 2x - 9 = 76$ to find x . *Ans.* $x = 5$.
4. Given $\frac{x^2}{2} - \frac{x}{3} + 7\frac{1}{5} = 20$ to find x . *Ans.* $x = 5.4093, \&c.$
5. Given $x^2 + x = a$ to find x .

Ans. $x = \sqrt{a + \frac{1}{4}} - \frac{1}{2}$

6. Given $x^2 + ax - b = c$ to find x .

Ans. $x = \sqrt{c + b + \frac{a^2}{4}} - \frac{a}{2}$

7. Given $x^2 - ax = -b$ to find x .

Ans. $x = \pm \sqrt{\frac{a^2}{4} - b} + \frac{a}{2}$

8. Given $\frac{ax^2}{b} - \frac{cx}{d} + \frac{e}{f} = \frac{g}{h}$ to find x .

Ans. $x = \pm \sqrt{\frac{b^2c^2}{4a^2d^2} + \frac{bg}{ab} - \frac{be}{af} + \frac{bc}{2ad}}$

9. Given $2x^4 - x^2 + 104 = 600$ to find x .

Ans. $x = 4$.

10. Given $3x^n - 2x^{\frac{n}{2}} - \frac{p}{9} = \frac{r}{9}$ to find x .

Ans. $x = \pm \sqrt{\frac{r+p+3}{27} + \frac{1}{3}}^{\frac{2}{n}}$

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QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whose difference is 8, and product 240.

*Let $x = 10$ the least number,
then will $x + 8 = 18$ the greater,
and $x \times x + 8 = x^2 + 8x = 240$ by the question;
Whence $x^2 + 8x + 16 = 240 + 16 = 256$ by completing the square,
also $x + 4 = \sqrt{256} = 16$ by evolution;
And therefore $x = 16 - 4 = 12 =$ lesser number, and $12 + 8 = 20 =$ greater.*

2. To divide the number 60 into two such parts that their product may be 864.

*Let $x =$ greater part,
then will $60 - x =$ lesser,
and $x \times 60 - x = 60x - x^2 = 864$ by the question,
that is $x^2 - 60x = -864$;
Whence $x^2 - 60x + 900 = -864 + 900 = 36$ by completing the square;
Also $x - 30 = \sqrt{36} = 6$ by extracting the root;
and therefore $x = 6 + 30 = 36 =$ greater part,
and $60 - x = 60 - 36 = 24 =$ lesser.*

3. Given the sum of two numbers $= 10$ (a), and the sum of their squares $= 58$ (b); to find those numbers.

*Let $x =$ greater of the two numbers,
then will $a - x =$ lesser;
And $x^2 + a - x^2 = 2x^2 + a^2 - 2ax = b$ by the question,*

QUADRATIC EQUATIONS. 87

or $x^2 + \frac{a^2}{2} - ax = \frac{b}{2}$ by division,

or $x^2 - ax = \frac{b}{2} - \frac{a^2}{2} = \frac{b-a^2}{2}$ by transposition.

Whence $x^2 - ax + \frac{a^2}{4} = \frac{b-a^2}{2} + \frac{a^2}{4} = \frac{2b-a^2}{4}$ by completing the square.

Also $x - \frac{a}{2} = \sqrt{\frac{2b-a^2}{4}}$ by extracting the root;

And therefore $x = \pm \sqrt{\frac{2b-a^2}{4}} + \frac{a}{2} = \text{greater number,}$

And $a - \frac{a}{2} \pm \sqrt{\frac{2b-a^2}{4}} = \mp \sqrt{\frac{2b-a^2}{4}} + \frac{a}{2} = \text{lesser.}$

Hence these two theorems being put into numbers give 7 and 3, for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth cost me; what was the price of the cloth?

Let $x =$ pounds the cloth cost,

Then $24 - x =$ whole gain,

But $100 : x :: x : 24 - x$ by the question,

Or $x^2 = 100 \times 24 - x = 2400 - 100x,$

That is $x^2 + 100x = 2400,$

Whence $x^2 + 100x + 2500 = 2400 + 2500 = 4900$ by completing the square,

And $x + 50 = \sqrt{4900} = 70$ by extraction of roots,

Consequently $x = 70 - 50 = 20 =$ price of the cloth.

5. A person bought a number of oxen for 80l. and if he had bought 4 more for the same money, he would have paid 1l. less for each; how many did he buy?

33 QUADRATIC EQUATIONS.

Suppose he bought x oxen,

Then $\frac{80}{x}$ = price of each,

And $\frac{80}{x+4}$ = price of each if $x+4$ had cost 80l.

But $\frac{80}{x} = \frac{80}{x+4} + 1$ by the question,

Or $80 = \frac{80x}{x+4} + x$,

Or $80x + 320 = 80x + x^2 + 4x$,

That is $x^2 + 4x = 320$,

Whence $x^2 + 4x + 4 = 320 + 4 = 324$ by completing the square,

And $x + 2 = \sqrt{324} = 18$ by evolution,

Consequently $x = 18 - 2 = 16$ = number of oxen required.

6. What two numbers are those, whose sum, product, and difference of their squares are all equal to each other?

Let x = greater number,

And y = lesser,

Then $\begin{cases} x+y=xy \\ x+y=x^2-y^2 \end{cases}$ by the question,

And $1 = \frac{x^2-y^2}{x+y} = x-y$, or $x = y+1$ from the 2d. equation.

Also $y+1+y = y+1 \times y$ from the first equation,

Or $2y+1 = y^2+y$,

That is $y^2-y=1$,

Whence $y^2-y+\frac{1}{4} = 1\frac{1}{4}$ by completing the square,

Also $y - \frac{1}{2} = \sqrt{1\frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ by evolution,

Consequently $y = \frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2}$, and $x = y + 1$
 $= \frac{\sqrt{5}+3}{2}$.

7. There are four numbers in arithmetical progression, whereof the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x =$ lesser extreme,
 and $y =$ common difference,

Then $x, x+y, x+2y, x+3y$ will be the four numbers,

and $\left\{ \begin{array}{l} x \times x + 3y = x^2 + 3xy = 45 \\ x + y \times x + 2y = x^2 + 3xy + 2y^2 = 77 \end{array} \right\}$ by the question,

Whence $2y^2 = 77 - 45 = 32$, and $y^2 = \frac{32}{2} = 16$ by subtraction and division,

Or $y = \sqrt{16} = 4$ by evolution,

Therefore $x^2 + 3xy = x^2 + 12x = 45$ by the 1st. equation,

Also $x^2 + 12x + 36 = 45 + 36 = 81$ by completing the square,

And $x + 6 = \sqrt{81} = 9$ by the extraction of roots,

Consequently $x = 9 - 6 = 3$, and the numbers are 3, 7, 11, and 15.

8. To find three numbers in geometrical progression, whose sum shall be 14, and the sum of their squares 84.

Let x, y , and z be the numbers sought,

Then $xz = y^2$ by the nature of proportion,

and $\left\{ \begin{array}{l} x + y + z = 14 \\ x^2 + y^2 + z^2 = 84 \end{array} \right\}$ by the question,

But $x + z = 14 - y$ by the 2d. equation,

90 QUADRATIC EQUATIONS.

and $x^2 + 2xz + z^2 = 196 - 28y + y^2$ by squaring both sides,

Or $x^2 + z^2 + 2y^2 = 196 - 28y + y^2$ by putting $2y^2$ for its equal $2xz$,

That is $x^2 + z^2 + y^2 = 196 - 28y$ by subtraction,

or $196 - 28y = 84$ by equality,

Hence $y = \frac{196 - 84}{28} = 4$ by transposition and division.

Again, $xz = y^2 = 16$, or $x = \frac{16}{z}$ by the 1st equation,

And $x + y + z = \frac{16}{z} + 4 + z = 14$ by the 2d equation,

Or $16 + 4z + z^2 = 14z$, or $z^2 - 10z = -16$,

Whence $z^2 - 10z + 25 = 25 - 16 = 9$ by completing the square,

And $z - 5 = \sqrt{9} = 3$, or $z = 3 + 5 = 8$,

Consequently $x = 14 - y - z = 14 - 4 - 8 = 2$, and the numbers are 2, 4, 8.

9. The sum (s) and the product (p) of any two numbers being given; to find the sum of the squares, cubes, biquadrates, &c. of those numbers.

Let the two numbers be denoted by x and y ;

Then will $\begin{cases} x + y = s \\ xy = p. \end{cases}$ by the question.

But $(x + y)^2 = x^2 + 2xy + y^2 = s^2$ by involution;

and $x^2 + 2xy + y^2 - 2xy = s^2 - 2p$ by subtraction,

that is $x^2 + y^2 = s^2 - 2p =$ sum of the squares.

Again, $x^2 + y^2 \times x + y = s^2 - 2p \times s$ by multiplication,

or $x^3 + xy \times x + y + y^3 = s^3 - 2sp$,

or $x^3 + sp + y^3 = s^3 - 2sp$ by substituting sp for its equal

$xy \times x + y$;

and therefore $x^3 + y^3 = s^3 - 3sp =$ sum of the cubes.

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In like manner, $\overline{x^3+y^3} \times \overline{x+y} = \overline{s^3-3sp} \times s$ by multiplication,

$$\text{or } x^4+xy \times \overline{x^2+y^2} + y^4 = s^4 - 3s^2p,$$

or $\overline{x^4+p \times s^2-2p+y^4} = \overline{s^4-3s^2p}$ by substituting $p \times s^2-2p$ for its equal $xy \times \overline{x^2+y^2}$;

And consequently, $x^4+y^4 = s^4-3s^2p-p \times s^2-2p = s^4-4s^2p+2p^2 = \text{sum of the biquadrates, or fourth powers; and so on for any power whatever.}$

10. The sum (a) and the sum of the squares (b) of four numbers, in geometrical progression, being given; to find those numbers.

Let x and y denote the two means, then will $\frac{x^2}{y}$ and $\frac{y^2}{x}$ be the two extremes, by the nature of proportion.

Also, let the sum of the two means $= s$, and their product $= p$.

And then will the sum of the two extremes $= a-s$ by the question, and their product $= p$, by the nature of proportion.

$$\text{Hence } \left\{ \begin{array}{l} x^2+y^2 = s^2-2p \\ \frac{x^4}{y^4} + \frac{y^4}{x^4} = \frac{(a-s)^2}{p^2} - 2p \end{array} \right\} \text{ by the last problem,}$$

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2} = s^2+\frac{(a-s)^2}{p} - 4p = b$ by the question.

$$\text{Again, } \frac{x^2}{y} + \frac{y^2}{x} = a-s \text{ by the question,}$$

$$\text{or } x^3+y^3 = xy \times a-s = p \times a-s.$$

But $x^3+y^3 = s^3-3sp$ by the last problem;

and therefore $p \times a-s = s^3-3sp$ by equality,

$$\text{or } pa-ps+3ps=pa+2ps=s^3,$$

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$$\text{or } p = \frac{s^3}{a+2s};$$

Whence $s^2 + a - s \Big| ^2 - 4p = s^2 + a - s \Big| ^2 - \frac{4s^3}{a+2s} = b$, by substitution,

$$\text{or } s^2 + \frac{b}{a} s = \frac{a^2 - b}{2} \text{ by reduction.}$$

And $s = \sqrt{\frac{a^2 - b}{2}} + \frac{b^2}{4a^2} \frac{b}{2a}$ by comp. the square, and extracting the root.

And from this value of s , all the rest of the quantities p , x , and y may be readily determined.

QUESTIONS FOR PRACTICE.

1. What two numbers are those whose sum is 20, and their product 36? *Ans. 2 and 18.*
2. To divide the number 60 into two such parts, that their product may be to the sum of their squares in the ratio of 2 to 5. *Ans. 20 and 40.*
3. The difference of two numbers is 3, and the difference of their cubes is 117: what are those numbers? *Ans. 2 and 5.*
4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was settled, two of them sneaked off, and then those who remained had 10s. a-piece more to pay than before: how many were there in company? *Ans. 7.*
6. A grazier bought as many sheep as cost him 60l. and, after reserving 15 out of the number, he sold the remainder for 54l. and gained 2s. a-head by them; how many sheep did he buy? *Ans. 75.*
7. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number; what are those numbers? *Ans. 3 and 18.*

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8. A person bought cloth for 33 l. 15 s. which he sold again at 2 l. 8 s. *per* piece, and gained by the bargain as much one piece cost him; required the number of pieces? *Ans.* 15.

9. What number is that, which when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Ans.* 24.

10. What two numbers are those, whose sum multiplied by the greater is equal to 77; and whose difference multiplied by the lesser is equal to 12? *Ans.* 4 and 7.

11. When will the hour, minute, and second hands of a clock be all together next after 12 o'clock? *Ans.* Only at 12 o'clock.

12. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers? *Ans.* 3 and 5.

13. The sum of two numbers is 7, and the sum of their 4th powers is 641; what are the numbers? *Ans.* 2 and 5.

14. The sum of two numbers is 6, and the sum of their 5th powers is 1056; what are the numbers? *Ans.* 2 and 4.

15. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers? *Ans.* 8, 12, 16, and 20.

16. To find four numbers in geometrical progression whose sum is 15, and the sum of their squares 85? *Ans.* 1, 2, 4, and 8.

17. Given $\sqrt{x^2 - \frac{a^4}{x^2}}^{\frac{1}{2}} + a^2 \sqrt{\frac{a^4}{x^2}}^{\frac{1}{2}} = \frac{x^2}{a}$, to find the value of x .

$$\text{Ans. } x = \frac{1}{2}a^2 + \sqrt{\frac{5a^4}{4}}$$

Of the NATURE and FORMATION of EQUATIONS in general.

All equations of superior orders are generated by the multiplication of equations of inferior orders, involving the same unknown quantity.

Thus, a *quadratic equation* is formed by the multiplication of two simple equations.

A *cubic equation* is produced by the continued multiplication of three simple equations; or from one quadratic and one simple equation.

A *biquadratic equation* is generated by the continued multiplication of four simple equations, or of two quadratic equations; or of one cubic and one simple equation, &c.

For, suppose the unknown quantity to be x , and it values in any simple equation to be a, b, c, d , &c.

Then those simple equations, by bringing all the terms to one side, will become $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, &c.

And the product of any two of these, as $\overline{x-a} \times \overline{x-b} = 0$, will give a *quadratic equation*, or one of two dimensions.

The product of any three of them, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$, will give a *cubic equation*, or one of three dimensions.

The product of any four of them, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$, will give a *biquadratic equation*, or one of four dimensions, &c.

From hence it appears, that every equation has as many roots as it has simple equations that produce it, or as there are units in the highest dimension of the unknown quantity.

For if any of the values of x ($a, b, c,$ or $d,$) be substituted in the place of x in this biquadratic equation $x-a \times x-b \times x-c \times x-d$, then will all the terms of that equation vanish, and the whole will be found equal to nothing. But as there are no other quantities, besides these four, that substituted in the place of x will made the product vanish; therefore the equation cannot possibly have more than four roots, or admit of more than four solutions.

And after the same manner it appears, that no equation can have more roots than it contains dimensions of the unknown quantity.

To make this still plainer, by an example in numbers; suppose the equation to be resolved be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that you discover that this equation is the same with the product of $x-1 \times x-2 \times x-3 \times x-4$.

Then you will certainly infer that the four values of x are 1, 2, 3, and 4; for any of these numbers being put for x will make that product, and consequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the proposed equation.

And it is certain that there can be no other values of x besides these four; since if any other number be substituted for x in those factors, there will none of them vanish, and therefore their product cannot be equal to nothing, according to the equation.

The roots of equations are either *positive* or *negative*, according as the roots of the simple equations from whence they are produced are positive or negative.

Thus, if you suppose $x=-a, x=b, x=-c,$ and $x=d$, then will $x+a=0, x-b=0, x+c=0,$ and $x-d=0$, and the equation $x+a \times x-b \times x+c \times x-d=0$ will have its roots $-a, +b, -c, +d$.

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The *signs*, and *co-efficients* of equations will be best understood by considering the following table; where the simple equations $x-a$, $x-b$, &c. are multiplied continually together, and produce, successively, the higher equations.

$$x-a=0$$

$$x-b=0$$

$$\left. \begin{array}{l} x^2-ax \\ -bx+ba \end{array} \right\} = 0, \text{ a quadratic,}$$

$$\times x-c=0$$

$$\left. \begin{array}{l} x^3-a \\ -b \end{array} \right\} x^2 + \left. \begin{array}{l} +ab \\ +ac \end{array} \right\} x - abc = 0, \text{ a cubic,}$$

$$\times x-d=0$$

$$\left. \begin{array}{l} x^4-a \\ -b \\ -c \\ -d \end{array} \right\} x^3 + \left. \begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \end{array} \right\} x^2 + \left. \begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array} \right\} x + abcd = 0, \text{ a biquadratic.}$$

&c.

From the inspection of these equations it is plain, that the co-efficient of the first term is *unity*.

The co-efficient of the second term is the *sum of all the roots* (a, b, c, d), *with contrary signs*.

The co-efficient of the third term is *equal to the sum of the rectangles of those roots, or of all the products that can possibly arise by combining them, two and two*.

The co-efficient of the fourth term is *equal to the sum of all the products that can possibly arise by combin-*

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ing them, three and three; and so on for any other co-efficient whatever.

The last term is always equal to the product of all the roots with contrary signs; and this reasoning will hold, whether the roots be positive or negative.

It likewise appears, from inspection, that the signs of all the terms of any equation in the table are alternately $+$ and $-$.

The first term is always some pure power of x , and is positive.

The second term is some power of x , multiplied by the quantities $-a$, $-b$, $-c$, &c. and since these quantities are all negative, it follows that the term itself must be negative also.

The third term has the product of any two of these quantities ($-a$, $-b$, $-c$,) for its co-efficient, and is therefore positive; since $- \times -$, as well as $+ \times +$, gives $+$, or an affirmative quantity.

For the same reason, the next co-efficient, which is formed of the products of any three of these negative quantities, must be negative; and the next following co-efficient, being made up of the products of any four of the said negative quantities, must be positive, and so on.

And from this reasoning it plainly appears that when all the roots are positive, the signs are plus and minus alternately.

But if the roots are all negative ($x = -a$, $x = -b$, $x = -c$, $x = -d$) then $x + a \times x + b \times x + c \times x + d = 0$, will express the equation to be produced; and all the terms will, plainly, be positive.

So that, when all the roots of an equation are negative, it is plain that there can be no changes in the signs of the terms of that equation.

And, in general, there will be as many positive roots in any equation, as there are changes in the

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signs of the terms of that equation, from + to -, or from - to +; and all the rest of the roots will be negative.

From this rule it follows that, in quadratic equations, the two roots may be either both positive, or both negative, or one negative and one affirmative.

Thus, in the equation, $x^2 - ax - bx + ab = 0$, $(x - a \times x - b)$ there are two changes of the signs, and therefore the roots are both positive.

In the equation $x^2 + ax + bx + ab = 0$, $(x + a \times x + b)$ there is no change of the signs, and consequently they are both negative.

And, in this equation, $x^2 - ax + ab = 0$, $(x - a \times x + b)$ one of the roots will be affirmative and one negative; for since the first term is positive and the last negative, there can be but one change in the signs, whether the second term be + or -.

In cubic equations, the roots may be all positive, or all negative; or two of them may be negative and one positive, or one negative and two positive.

Thus, in the equation $x - a \times x - b \times x - c = 0$, the signs will be alternately + and -; and, as the number of changes is three, the roots must be all positive.

In the equation $x + a \times x + b \times x + c = 0$, where there are no changes of the signs, the roots must be all negative.

In the equation $x^3 - a - b + cx^2 + ab - ac - bcx + abc = 0$, $(x - a \times x - b \times x + c)$ the number of changes will be two, and consequently two of the roots will be affirmative and one negative.

For if $a + b$ is greater than c , the second term must be negative, its co-efficient being $-a$, $-b$, $+c$;

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and if $a+b$ is less than c , the third term must be negative, its co-efficient $+ab-ac-bc$ ($ab-c \times a+b$) being in that case negative.

In the equation $x^3 + a + b - cx^2 + ab - ac - bcx - abc = a$, there can be only one change of the signs, and therefore one of the roots is positive, and the other two negative.

For if $a+b$ is less than c then the second term is negative, and the third must be negative also; and if $a+b$ is greater than c the second term will be positive, and there can be but one change in the other two terms, whatever may be their signs.

And, in the same manner, this reasoning may be extended to equations of higher dimensions, and therefore the rule will be found to be true in all kinds of equations whatever.

PROBLEM I.

To increase or diminish the roots of an equation by any given quantity.

RULE.

1. Take some new letter, and connect it with the given quantity by the signs — or +, according as it is required to be increased or diminished.

2. Substitute the powers of this quantity in the equation, instead of the powers of the unknown letter, and there will arise a new equation, whose roots will be augmented or diminished as required.

EXAMPLES:

1. Let the quadratic equation $x^2 - 8x + 15 = 0$, be given; it is required to increase its roots by 7.

Suppose $x = y - 7$,

Then $x^2 = y^2 - 14y + 49$

$8x = 8y - 56$

$15 = 15$

$y^2 - 6y + 8 = 0$ = equal to the equation required*.

2. Let $x^3 - px^2 + qx - r = 0$, be the equation given; it is required to diminish the roots by the quantity e .

Suppose $x = y + e$;

Then $x^3 = y^3 + 3ey^2 + 3e^2y + e^3$
 $-px^2 = -py^2 - 2pey - pe^2$
 $+qx = +qy + qe$
 $-r = -r$

$= 0$, the new equation required†.

3. Let $x^3 + x^2 - 10x + 8 = 0$ be given, and let its roots be increased by 4.

* For in the former equation $x^2 + 8x + 15 = 0$, the roots are -3 and -5 , and in the equation $y^2 - 6y + 8 = 0$, the roots are 2 and 4 ; and therefore the difference is 7 , as required.

† The last term of this transformed equation, is the same as the equation given, having e in the place of y .

Suppose $x=y-4$;

Then $x^2=y^2-12y^2+48y-64$

$$+ x^2 = + y^2 - 8y + 16$$

$$-10x = -10y + 40$$

$$+ 8 = + 8$$

$$y^3 - 11y^2 + 30y = 0, \text{ or } y^2 - 11y + 30 = 0, \text{ the equation required.} \dagger$$

PROBLEM II.

To take away the second term from any equation.

RULE.

1. Divide the co-efficient of the second term by the index of the highest power of the unknown quantity.

2. Annex the quotient, with its sign changed, to some new letter, and this being substituted for its equal in the given equation, will destroy the second term, as required.

† In this example the given equation is reduced to a quadratic, and in the present case, as well as in all others where the last term vanishes, the number assumed (-4) is one of the roots of the proposed equation.

The affirmative roots of an equation are changed into negative ones of the same value, and the negative roots into affirmative ones, by only changing the signs of the terms alternately, beginning with the second.

EXAMPLES:

1. Let the quadratic equation $x^2 - 8x + 15 = 0$ be given; it is required to take away its second term.

Suppose $x = y + 4(y + \frac{8}{2})$;

Then $x^2 = y^2 + 8y + 16$

$- 8x = -8y - 32$

$+ 15 = + 15$

$y^2 - 1 = 0 = \text{equation required}^*.$

2. Let the equation $x^3 - 9x^2 + 26x - 34 = 0$ be given; it is required to exterminate its second term.

Suppose $x = y + 3(y + \frac{9}{3})$;

Then $x^3 = y^3 + 9y^2 + 27y + 27$

$- 9x^2 = -9y^2 - 54y - 81$

$+ 26x = + 26y + 78$

$- 34 = - 34$

$y^3 - y - 10 = 0 = \text{equation required}^*.$

Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, are $+1, +2, +3, -5$; but by changing only the second and fourth terms, the equation becomes $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, and the roots are $-1, -2, -3, +4$.

All the roots of an equation may also be made affirmative or negative, by increasing or diminishing each of them by some known quantity.

From this example it appears, that any quadratic equation may be resolved without completing the square, by only taking away the second term; for since $y^2 = 1$, or $y = \pm 1 = 1$, we shall have $x = y + 4 = 1 + 4 = 5$, the root required.

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3. Let $x^4 + 8x^3 - 5x^2 + 10x - 4 = 0$ be given, to exterminate the second term.

Suppose $x = y - z(y - \frac{3}{4})$;

Then $x^4 = y^4 - 8y^3 + 24y^2 - 32y + 16$

+ $8x^3 = +8y^3 - 48y^2 + 96y - 64$

- $5x^2 = -5y^2 + 20y - 20$

+ $10x = +10y - 20$

- $4 = -4$

$y^4 - 29y^2 + 94y - 92 = 0 = \text{equation required.}$

4. Let $x^4 - px^3 + qx^2 - rx + s = 0$ be given, to exterminate the second term.

Suppose $x = y + \frac{p}{4}$;

Then $x^4 = y^4 + py^3 + \frac{3p^2y^2}{8} + \frac{p^3y}{16} + \frac{p^4}{256}$

- $px^3 = -py^3 - p^2y^2 - \frac{3p^3y}{16} - \frac{p^4}{16}$

+ $qx^2 = +qy^2 + \frac{pqy}{2} + \frac{qp^2}{16}$

- $rx = -ry - \frac{rp}{4}$

+ $s = +s$

$y^4 + \frac{8q - 5p^2}{8}y^2 + \frac{8pq - 2p^3 - 16r}{16}y + \frac{-15p^4 + 16qp^2 - 64rp + 256s}{256} = 0 = \text{equation required.}$

P R O B L E M III.

To find whether some, or all the roots of an equation be rational; and, if so, what they are.

R U L E . *

1. Find all the divisors of the last term, and substitute them one by one for the unknown quantity.
2. Then, if the positive and negative terms destroy each other, the divisor, so substituted, will be one of the roots of the equation.
3. But, if none of the divisors succeed, the roots are, for the general part, either irrational or impossible.

Note, When the divisors of the last term are too numerous, they may be diminished by changing the equation into another, whose roots are augmented or decreased by an unit, or some other known quantity.

E X A M P L E S :

1. Let $x^3 - 4x^2 - 7x + 10 = 0$ be the equation proposed.

Then the divisors of (10) the last term will be +1, -1, +2, -2, +5, -5, +10, -10.

* Since the last term, in any equation, is always equal to a multiple of all the roots in that equation, those roots must, therefore, necessarily be found in the number of its divisors.

And these being substituted successively instead of x , will give

$$\begin{array}{rcl} 1-4-7+10 & = & 0 \\ -1-4+7+10 & = & 12 \\ 8-16-14+10 & = & -12 \\ -8-16+14+10 & = & 0 \\ 125-100-35+10 & = & 0 \end{array}$$

Therefore $+1-2$ and $+5$ are the three roots of the equation required.

2. Let $y^4-4y^3-3y+32=0$, be the equation proposed.

1. Change it into another, the number of whose divisors shall be less; thus:

Suppose $y=x+1$;

Then $y^4=x^4+4x^3+6x^2+4x+1$

$$-4y^3 = -4x^3-12x^2-12x-4$$

$$-8y = -8x-8$$

$$+32 = +32$$

$$x^4-6x^2-16x+21=0 = \text{new equation.}$$

2. The divisors of the last term (21) of this new equation are

$$1, -1, +3, -3, +7, -7, +21, -21.$$

And if these be substituted successively instead of x , we shall have

$$1-6-16+21=0$$

$$1-6+16+21=32$$

$$81-54-48+21=0$$

$$81-54+48+21=96$$

&c. where none of the others succeed.

So that 1 and 3 are the only rational roots, the other two being impossible.

3. Let $x^3+3ax^2-4a^2x-12a^3=0$, be the equation proposed.

Here the numeral divisors of the last term ($12a^3$) are

$$1, -1, +2, -2, +3, -3, +4, -4, +6, -6, +12, -12.$$

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And by substituting these successively instead of x , we shall have

$$\begin{array}{r} 1 + 3 - 4 - 12 = -12 \\ -1 + 3 + 4 - 12 = -6 \\ 8 + 12 - 8 - 12 = 0 \\ -8 + 12 + 8 - 12 = 0 \\ 27 + 27 - 12 - 12 = 30 \\ -27 + 27 + 12 - 12 = 0 \end{array}$$

Therefore the three roots are $2a$, $-2a$ and $3a$.

PROBLEM IV.

To discover the roots of equations by SIR ISAAC NEWTON'S method of divisors.

RULE.

1. Instead of the unknown quantity, substitute successively three, or more, terms of the arithmetical progression $2, 1, 0, -1, -2$.
2. Collect all the terms of the equation into one sum, and place them, together with their divisors, in a perpendicular line, right against the corresponding terms of the progression $2, 1, 0, -1, -2$.
3. Seek amongst the divisors for an arithmetical progression whose terms correspond with the order of the terms $2, 1, 0, -1, -2$, and whose common difference is either an unit, or some divisor of the co-efficient of the highest power of the unknown quantity in the given equation.
4. Divide that term of the progression, thus found, which stands against the term 0 in the first progression by the ratio or common difference.
5. To the quotient, last found, prefix the sign $+$ or $-$, according as the progression is increasing or

decreasing, and this number being substituted for the unknown quantity, will give one of the roots of the equation.

EXAMPLES:

1. Let $x^3 - x^2 - 10x + 6 = 0$, be the equation proposed.

Then, by substituting successively the terms of the progression 2, 1, 0, -1, instead of x , the work will stand as follows:

1st prog.	results.	divisors.	2d prog.
2	-10	1.2.5.10	5
1	-4	1.2.4	4
0	+6	1.2.3.6	3
-1	+14	1.2.7.14	2

And -3, the term standing against 0, being substituted for x , gives $-27 - 9 + 30 + 6 = 0$; and therefore -3 is a root of the equation.

2. Let $2x^3 - 5x^2 + 4x - 10 = 0$, be the equation proposed.

Then, by substituting successively the terms of the progression 2, 1, 0, -1, -2, instead of x , the work will stand as follows:

1st prog.	results.	divisors.	2d prog.
2	-6	1.2.3.6	1
1	-9	1.3.9	3
0	-10	1.2.5.10	5
-1	-21	1.3.7.21	7
-2	-54	1.2.3.6.9	9

Here 5, the term standing against 0, being divided by 2, the common difference, gives $2\frac{1}{2}$; and this being substituted for x , gives $31\frac{1}{4} - 31\frac{1}{4} + 10 - 10 = 0$; and therefore $2\frac{1}{2}$ is a root of the equation.

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3. Let $x^4 + x^3 - 29x^2 - 9x + 180 = 0$, be the equation proposed.

Then, by substituting as before, the work will stand as follows :

1st pro. results.	divisors.	progressions.
2	70	1. 2. 5. 7. 10. 14 $\mathcal{E}c$
1	144	1. 2. 3. 4. 6. 8 $\mathcal{E}c$
0	180	1. 2. 3. 4. 5. 6 $\mathcal{E}c$
-1	160	1. 2. 4. 5. 8. 10 $\mathcal{E}c$
-2	90	1. 2. 3. 5. 6. 9 $\mathcal{E}c$

So that here are four progressions, and the numbers 3, 4, -3, and -5, being each substituted for x , make the whole equation vanish, and are therefore the roots required.

PROBLEM V.

To find the roots of cubic equations, according to the method of CARDAN.

RULE.*

1. Take away the second term of the equation, by problem 2d, and it will be reduced to this form $x^3 \pm ax = \pm b$.

* The rule, from whence this method is derived, is $x = \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} + \sqrt[3]{\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$, and the investigation of it is as follows :

Let the equation, whose root is required, be $x^3 + ax = b$.

Put $y + z = x$, and $3yz = -a$.

Then, by substituting these values in the given equation we shall have, $y^3 + 3y^2z + 3yz^2 + z^3 + a \times y + z = y^3 + z^3 + 3yz \times y + z + a \times y + z = y^3 + z^3 - a \times y + z + a \times y + z = y^3 + z^3 = b$.

2. Substitute the values of a and b , with their proper signs, in the following expression, and it will give the root required. Thus:

$$\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}} = x = \text{root required.}$$

Note. When a is negative, and $\frac{1}{27}a^3$ is greater than $\frac{1}{4}b^2$, the equation, by this rule, is generally impossible.

And if, from the square of this last equation, there be taken 4 times the cube of the equation $yz = -\frac{1}{3}a$, we shall have $y^6 - 2y^3z^3 + z^6 = b + \frac{4}{27}a^3$, or $y^3 - z^3 = \sqrt{b + \frac{4}{27}a^3}$.

But the sum of this equation and $y^3 + z^3 = b$, is $2y^3 = b + \sqrt{b + \frac{4}{27}a^3}$; and their difference is $2z^3 = b - \sqrt{b + \frac{4}{27}a^3}$; and therefore $y = \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$, and $z =$

$$\sqrt[3]{\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}.$$

And from hence it appears, that $y + z$, or its equal x , is $= \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} + \sqrt[3]{\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$, which is the theorem.

Or, since z is $= -\frac{\frac{1}{3}a}{y}$, it will be $y + z = x =$

$$\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}}, \text{ the same as the rule.}$$

This method of solving cubic equations is usually ascribed to *Cardan*; but the invention is, undoubtedly, not his.—The authors of it were *Scipio Ferreus* and *Nicholas Tartalea*, who discovered it about the same time, independently of each other, as is proved by *M. de Montucla dans son Histoire des Mathématiques*.

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EXAMPLES:

1. Let $y^3 + 3y^2 + 9y = 13$, be the equation proposed; it is required to find the value of y .

1. In order to destroy the second term, let $y = x - 1$;
then

$$\begin{aligned} y^3 &= x^3 - 3x^2 + 3x - 1 \\ 3y^2 &= +3x^2 - 6x + 3 \\ 9y &= +9x - 9 \end{aligned}$$

$$\begin{aligned} x^3 + 6x - 7 &= 13 \\ \text{or } x^3 + 6x &= 20. \end{aligned}$$

2. For a put 6, and for b 20, and we shall have

$$\begin{aligned} x &= \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}} = \\ &= \frac{\sqrt[3]{10 + \sqrt{100 + 8}}}{\sqrt[3]{10 + \sqrt{100 + 8}}} - \frac{2}{\sqrt[3]{10 + \sqrt{100 + 8}}} = \\ &= \frac{\sqrt[3]{10 + 10.3923}}{\sqrt[3]{10 + 10.3923}} - \frac{2}{\sqrt[3]{10 + 10.3923}} = \sqrt[3]{20.3923} - \\ &= \frac{2}{2.732} = 2.732 - \frac{2}{2.732} = 2.732 - .732 = 2; \text{ that} \\ &\text{is } x = 2, \text{ and consequently } y = 1 = \text{root required.} \end{aligned}$$

2. Given $x^3 - 6x = -9$, to find the value of x .

Here $a = -6$, and $b = -9$; and therefore we shall have

$$\begin{aligned} x &= \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}} = \\ &= \frac{\sqrt[3]{-4\frac{1}{2} + \sqrt{20\frac{1}{4} - 8}}}{\sqrt[3]{-4\frac{1}{2} + \sqrt{20\frac{1}{4} - 8}}} - \frac{-2}{\sqrt[3]{-4\frac{1}{2} + \sqrt{20\frac{1}{4} - 8}}} = \end{aligned}$$

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$$\sqrt[3]{-4\frac{1}{2} + 3\frac{1}{2}} - \frac{-2}{\sqrt[3]{-4\frac{1}{2} + 3\frac{1}{2}}} = \sqrt[3]{-1} - \frac{-2}{\sqrt[3]{-1}} = -1 - \frac{-2}{-1} = -1 - 2 = -3; \text{ that is } x = -3 = \text{root required.}$$

EXAMPLES FOR PRACTICE.

1. Given $x^3 - 5x^2 + 10x - 8 = 0$, to find x . *Ans. x = 4.*
2. Given $x^3 + 30x = 117$, to find x . *Ans. x = 3.*
3. Given $y^2 - 36y = 91$, to find y . *Ans. y = 7.*
4. Given $x^3 + 6x = 30\sqrt{3}$, to find x . *Ans. x = 2\sqrt{3}.*
5. Given $y^3 + 24y = 250$, to find y . *Ans. y = 5.05.*
6. Given $y^6 - 3y^4 - 2y^2 - 8 = 0$, to find y . *Ans. y = 2.*

PROBLEM VI.

To find the roots of biquadratic equations, according to the method of DES CARTES.

RULE.

1. Take away the second term of the equation by problem 2, and it will be reduced to the form $x^4 + qx^2 + rx + s = a$.

* *Investigation of the rule.* Let the given equation $x^4 + qx^2 + rx + s = 0$, be equal to the product of the two quadratic equations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0 = x^4 + f + g - e^2 x^2 + eg - ef x + fg = 0$.

Then, by equating the homologous terms, we shall have $f + g - e^2 = q$, $eg - ef = r$, and $fg = s$; and therefore $f =$

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2. From the cubic equation $y^3 + 2qy^2 + \overline{q^2 - 4y - r^2} = 0$ take the second term, and find the value of y by the last problem.

3. Put $e = \sqrt{y}$, $f = \frac{q}{2} + \frac{e^2}{2} - \frac{r}{2e}$, and $g = \frac{q}{2} + \frac{e^2}{2} + \frac{r}{2e}$.

4. Find the roots of the two quadratic equations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$, and they will be the four roots of the biquadratic required.

EXAMPLES:

1. Let $x^4 - 4x^3 - 8x + 32 = 0$, be the equation proposed, in which it is required to find the value of x .

$$\frac{q}{2} + \frac{e^2}{2} - \frac{r}{2e}, g = \frac{q}{2} + \frac{e^2}{2} + \frac{r}{2e}, \text{ and } s = f \times g = \frac{q^2}{4} + \frac{qe^2}{2} + \frac{e^4}{4} - \frac{r^2}{4e^2}.$$

And from this last equation we shall have $e^6 + 2qe^4 + q^2 - 4s \times e^2 = r^2 = 0$ to a cubic equation, in which the value of e may be found, as in the last problem.

But $f \left(= \frac{q}{2} + \frac{e^2}{2} - \frac{r}{2e} \right)$ and $g \left(= \frac{q}{2} + \frac{e^2}{2} + \frac{r}{2e} \right)$ are also known; and therefore the roots of the quadratic equations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$, may be determined, and are the four roots of the biquadratic equation required.

Q. E. I.

Note, The co-efficient of x is put equal to e , in both the equations, because when the second term is wanting, the sum of the positive roots is always equal to the sum of the negative ones, with a contrary sign.

This rule has sometimes been ascribed to *Des Cartes*, and sometimes to *Bombelli*, an Italian; but the original inventor of it was *Louis Ferrari*.

1. To take away the second term, let $x+1=x$; then

$$\begin{array}{r} x^4 = x^4 + 4x^3 + 6x^2 + 4x + 1 \\ -4x^3 = -4x^3 - 12x^2 - 12x - 4 \\ -8x = -8x - 8 \\ +32 = +32 \end{array}$$

$$x^4 - 6x^2 - 16x + 21 = 0, \text{ or}$$

$$y^3 - 12y^2 - 48y - 256 = 0, \text{ for the cubic equation.}$$

2. To take away the second term from this equation, let $p+4=y$; then

$$\begin{array}{r} y^3 = p^3 + 12p^2 + 48p + 64 \\ -12y^2 = -12p^2 - 96p - 192 \\ -48y = -48p - 192 \\ -256 = -256 \end{array}$$

$$p^3 - 96p = 576$$

The following method of resolving biquadratic equations, is taken from Simpson's Algebra, page 150, 2d edition.

In the method of Des Cartes, above explained, all biquadratic equations are supposed to be generated from the multiplication of two quadratic ones: but according to the way which I am now going to lay down, every such equation is conceived to arise by taking the difference of two complete squares.

Here, the general equation $x^4 + ax^3 + bx^2 + cx + d = 0$ being proposed, we are to assume $\left[x^2 + \frac{1}{2}ax + A \right]^2 - Bx + C$ $= x^4 + ax^3 + bx^2 + cx + d$; in which A, B, and C, represent unknown quantities to be determined.

Then, $x^2 + \frac{1}{2}ax + A$, and $Bx + C$ being actually involved, will give $x^4 + ax^3 + 2Ax^2$

$$\left. \begin{array}{l} + \frac{1}{2}a^2x^2 + aAx + A^2 \\ - B^2x^2 - 2BCx - C^2 \end{array} \right\} = x^4 + ax^3 + bx^2$$

$+ cx + d$: from whence, by equating the homologous terms, we shall have

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3. To find the value of p , by CARDAN's rule for cubic equations.

$$\begin{aligned} & \sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}} = \\ & \frac{\sqrt[3]{288 + \sqrt{288^2 - 32^3}} - \frac{-32}{\sqrt[3]{288 + \sqrt{288^2 - 32^3}}}}{\sqrt[3]{288 + \sqrt{288^2 - 32^3}}} = \\ 12 = p; \text{ and therefore } y = 16, \text{ or } \sqrt{y} = 4, f = \frac{-6}{2} + \frac{16}{2} \\ + \frac{16}{8} = 7, \text{ and } g = \frac{-6}{2} + \frac{16}{2} - \frac{16}{8} = 3. \end{aligned}$$

4. To find the roots of the two quadratic equations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$.

1. $2A + \frac{1}{3}a^2 - B^2 = b$, or $2A + \frac{1}{3}a^2 - b = B^2$;
2. $aA - 2BC = c$, or $aA - c = 2BC$;
3. $A^2 - C^2 = d$, or $A^2 - d = C^2$.

Let now the first and last of these equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{3}$ of the square of the second; that is $2A + \frac{1}{3}a^2 - b \times A^2 - d = \frac{1}{3} \times a^2 A^2 - 2acA + c^2$;

Whence, by denoting the given quantities $\frac{1}{3}ac - d$, and $\frac{1}{3}c^2 + d \times \frac{1}{3}a^2 - b$ by k and l , respectively, there will arise this cubic equation, $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = 0$; by means whereof, the value of A may be determined; and therefore, from the preceding equations, both B and C will also be known, B being found from thence $= \sqrt{2A + \frac{1}{3}a^2 - b}$, and $C = \frac{aA - c}{2B}$.

The several values of A , B , and C being thus found, that of x will also be readily obtained: for $x^2 + \frac{1}{2}ax + A^2 - bx + C^2$ being universally, in all circumstances of x , equal to $x^2 + ax^3 + bx^2 + cx + d$, it is evident, that, when the value of x is taken such that the latter of these expressions becomes equal to nothing, the former must likewise be equal to nothing; and consequently $x^2 + \frac{1}{2}ax + A^2 = Bx + C^2$.

$$x^2 + ex + f = x^2 + 4x + 7 = 0$$

$$x^2 - ex + g = x^2 - 4x + 3 = 0$$

In the first of these $x = -2 + \sqrt{-3}$, or $-2 - \sqrt{-3}$,

And in the second $x = 3$, and 1.

Therefore 3, 1, $-2 + \sqrt{-3}$, and $-2 - \sqrt{-3}$ are the four roots of the equation $x^4 - 6x^2 - 16x + 21 = 0$.

And if unity be added to each of them, we shall have 4, 2, $-1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$ for the roots of $x^4 - 4x^3 - 8x + 32 = 0$, the equation proposed; the two last of which are impossible.

2. Given $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$, to find the values of x . *Ans.* $x = 1, 2, -3$ and -2 .

3. Given $x^4 - 25x^2 + 60x - 36 = 0$, to find the values of x . *Ans.* 1, 2, 3 and -6 .

4. Given $y^4 - 8y^3 + 14y^2 + 4y - 8 = 0$, to find the value of y .

Ans. $y = 3 + \sqrt{5}$, $3 - \sqrt{5}$, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.

And therefore, by extracting the square root of both sides of the equation, we shall have $x^2 + \frac{1}{2}ax + A = \pm Bx \pm C$; or $x = \pm \frac{1}{2}B - \frac{1}{4}a \pm \frac{1}{4}a \pm \frac{1}{2}B \pm \frac{1}{2}C - A \pm \frac{1}{4}a^2 \mp \frac{1}{4}aB + \frac{1}{4}B^2 \pm C - A$; which exhibits all the different roots of the given equation, according to the variation of the signs.

This method will be found to have many advantages over that given above. In the first place, there is no necessity of being at the trouble of exterminating the second term of the equation, in order to prepare it for a solution: secondly, the equation $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = 0$, here brought out, is of a more simple form than that derived from the former method: and thirdly, the value of A in this equation, will always be commensurate and rational; not only when all the roots of the given equation are commensurate, but also when they are irrational, and even impossible.

EXAMPLE. Let there be given $x^4 + 12x - 17 = 0$, to find the values of x .

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5. Given $x^4 + 12x - 17 = 0$, to find the values of x .

Ans. $x = \frac{1}{2}\sqrt{2} \pm \sqrt{-3\sqrt{2} - \frac{1}{2}}$, and $-\frac{1}{2}\sqrt{2} \pm \sqrt{3\sqrt{2} - \frac{1}{2}}$.

6. Given $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$, to find the values of x .

Ans. $x = \pm \frac{5}{2}\sqrt{3} + \frac{3}{2} \pm \sqrt{17 \pm \frac{21}{2}\sqrt{3}}$.

7. Given $x^4 - 3x^2 - 4x - 3 = 0$, to find the values of x .

Ans. $x = \frac{1 \pm \sqrt{3}}{2}$ and $\frac{-1 \pm \sqrt{-3}}{2}$.

Here, by comparing this with the general equation $x^4 + ax^3 + bx^2 + cx + d = 0$, we shall have $a = 0$, $b = 0$, $c = 12$, and $d = -17$; and therefore $k (= \frac{1}{3}ac - d) = 17$, $l (= \frac{1}{3}c^2 + d \times \frac{1}{3}a^2 - b) = 36$, and $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = A^3 + 17A - 18 = 0$.

And, from this equation, A will be found equal to 1; and therefore $B (= 2A + \frac{1}{3}a^2 - l)^{\frac{1}{2}} = \sqrt{2}$, $C (= \frac{aA - c}{2B}) = \frac{-12}{2\sqrt{2}} = -3\sqrt{2}$, and $x = \pm \frac{1}{2}\sqrt{2} \pm \frac{1}{2} \pm 3\sqrt{2} - 1)^{\frac{1}{2}} = \pm \frac{1}{2}\sqrt{2} \pm \sqrt{3\sqrt{2} - \frac{1}{2}}$.

Hence, the four roots of the equation are $\frac{1}{2}\sqrt{2} + -3\sqrt{2} - \frac{1}{2}$, $\frac{1}{2}\sqrt{2} - -3\sqrt{2} - \frac{1}{2}$, $-\frac{1}{2}\sqrt{2} + 3\sqrt{2} - \frac{1}{2}$, and $-\frac{1}{2}\sqrt{2} - 3\sqrt{2} - \frac{1}{2}$; the first and second of which are impossible.

In some particular cases of this rule, the roots may be found by means of quadratics only.

Several other methods of solving biquadratic equations have been invented by different authors; but one of the most ingenious is that given by *M. Euler*, in page 664 of his *Elements D'Algebre*.

Equations of five, or more dimensions, may be reduced to those of an inferior degree; but the process will be exceedingly tedious, as no general rule can be given for resolving them.

P R O B L E M VII.

To find the roots of equations in general, by the methods of approximation and converging series.

R U L E. *

1. Find, by trial, a number nearly equal to the root required.

2. Call the number, thus found, r , and let z be put equal to the difference between r and the true root x .

3. Instead of x , in the given equation, substitute its equal $r \pm z$, and there will arise a new equation, affected only with z and known quantities.

4. Reject all those quantities in which there are two or more dimensions of z , and the value of z will be found by means of a simple equation.

5. Add the value of z , thus found, to the value of r , and it will give the root required *nearly*.

6. If this root is not sufficiently near the truth, repeat the operation, by substituting it instead of r , in the equation exhibiting the value of z , and it will give a *second correction* for the root required.

E X A M P L E S :

1. Given $x^2 - 5x - 31 = 0$, to find the value of x by approximation.

* The rules hitherto given, for finding the roots of equations, are either very troublesome and laborious, or else confined to particular cases; but this method, by converging series, is universal, extending to all kinds of equations whatever; and, though not accurately true, gives the value sought to any assigned degree of exactness.

The method of obtaining the roots of equations, by approximation, was first made use of by *Vieta*.

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The root, found by trial, is nearly equal to 8;

Let, therefore, $8=r$, and $r+x=x$; then

$$\begin{aligned}x^3 &= r^3 + 3r^2x + 3rx^2 + x^3 \\-5x &= -5r - 5x \\-31 &= -31\end{aligned}$$

$$r^3 + 3r^2x - 5r - 5x - 31 = 0;$$

$$\text{or } x = \frac{31 + 5r - r^3}{2r - 5} = \frac{31 + 40 - 64}{16 - 5} = \frac{7}{11} = .6, \text{ and } x = 8.6 \text{ nearly.}$$

And, again, if 8.6 be substituted in the place of r in the last equation, we shall have

$$x = \frac{31 + 5r - r^3}{2r - 5} = \frac{31 + 43 - 73.96}{17.2 - 5} = \frac{0.04}{12.2} = .0032$$

and $x = 8.6 + .0032 = 8.6032$ nearly.

And, if this value be again substituted for r , it will give $x = .0000077808$, and $x = 8.603277808$; and so on to any degree of exactness required.

2. Given $x^3 + x^2 + x = 90$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 4;

Let, therefore, $4=r$, and $r+x=x$, then,

$$\begin{aligned}x^3 &= r^3 + 3r^2x + 3rx^2 + x^3 \\x^2 &= r^2 + 2rx + x^2 \\x &= r + x\end{aligned}$$

$$r^3 + 3r^2x + r^2 + 2rx + r + x = 90;$$

$$\text{or, } x = \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \frac{90 - 64 - 16 - 4}{48 + 8 + 1} = \frac{6}{57} = .10,$$

and $x = 4.1$ nearly.

And again, if 4.1 be substituted in the place of r in the last equation, we shall have

$$\frac{90-r^3-r^2-r}{3r^2+2r+1} = \frac{90-68.921-16.81-4.1}{50.43+8.2+1} = .00283,$$

and $x=4.1+.00283=4.10283$ nearly; and so on to any degree of exactness required.

3. Given $x^2+20x=100$, to find the value of x by approximation. *Ans.* $x=4.1421356236$.

4. Given $x^2+10x^2+5x=2600$, to find the value of x by approximation. *Ans.* 10.1794653 .

5. Given $x^3+2x^2-23x-70=0$, to find the value of x . *Ans.* $x=5.1349$.

6. Given $x^3-15x^2+63x-50=0$, to find the value of x . *Ans.* $x=1.02803923127$.

7. Given $x^4-3x^2-75x=10000$, to find the value of x . *Ans.* $x=9.8860027$.

8. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x . *Ans.* $x=8.1414455$.

R U L E II.

1. Assume the general equation $ax+bx^2+cx^3+dx^4$, &c. $=p$; where x is the converging quantity, and a, b, c, d , &c. co-efficients whose values are known.

2. Then will $\frac{ap}{a^2+bp}$ be an approximation of the first degree.

3. And, if s be put $=\frac{b}{a}-\frac{c}{b}$ we shall have $\frac{a+sp \times p}{a^2+b+as \times p}$ for an approximation of the second degree.

4. And, in like manner, if w be put $=\frac{2b}{a}+$

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$\frac{ad-bc}{b^2-ac}$, then will $\frac{ap \times a + wp}{a \times a^2 + b + aw \times p + w - s \times p^2}$ be
an approximation of the third degree, &c.

EXAMPLES:

1. Given $x^2 + 20x = 100$, to find the value of x .

The root, found by trial, is nearly equal to 4;

Let therefore $4 + z = x$, and, by substitution, the equation will become $28z + z^2 = 4$.

Whence, from the rule, $a = 28$, $b = 1$, $c = 0$, &c. and $p = 4$.

Therefore $z = \frac{ap}{a^2 + bp} = \frac{112}{788} = \frac{28}{197} = .14213$ for the first approximation.

And, since $s = \frac{b}{a} - \frac{c}{b} = \frac{1}{28}$, we shall have

$$z = \frac{a + sp \times p}{a^2 + b + as \times p} = \frac{28 + \frac{1}{7} \times 4}{28 \times 28 + 1 + 1 \times 4} = \frac{28 + \frac{1}{7}}{28 \times 7 + 2} = \frac{197}{1386} = .14213564 \text{ for the second approximation.}$$

And, in like manner, since $w = \frac{2b}{a} + \frac{ad-bc}{b^2-ac} = \frac{1}{14}$, it

$$\text{will be } z = \frac{ap \times a + wp}{a \times a^2 + b + aw \times p + w - s \times p^2}$$

$$\frac{28 \times 4 \times 28 + \frac{2}{7}}{28 \times 784 + 12 + \frac{2}{7}} = \frac{28 \times 28 + \frac{2}{7}}{7 \times 796 + \frac{2}{7}} = \frac{28 \times 198}{49 \times 796 + 1} =$$

$$\frac{5544}{39005} = .1421356236; \text{ or } x = 4.1421356236 \text{ for the}$$

root required, extremely near.

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2. Given $x^2 + 4x - 10 = 100$, to find the value of x . *Ans.* 10.677078.

3. Given $x^3 - 17x^2 + 54x = 350$, to find the value of x . *Ans.* 14.95407.

4. Given $x^3 - 2x - 5 = 0$, to find the value of x . *Ans.* 2.09455147.

5. Given $2y^4 - 16y^3 + 40y^2 - 30y = -1$, to find the value of y . *Ans.* $y = 1.284724$.

PROBLEM VIII.

To extract the root of any pure power in numbers.

R U L E.

1. Let $m =$ the number whose root is required;
 $r =$ nearest root which can be found by trial; and
 $n =$ to the index.

2. Then, by putting $v = \frac{nr^n}{m - r^n}$, we shall have

$$x = r + \frac{r \times 6v + n + 1}{v \times 6v + 4n - 2} = \text{root, nearly; or } x = r +$$

$$\frac{r \times 2v + n}{v \times 2v + 2n - 1 + \frac{1}{6} \times n - 1 \times 2n - 1} \text{ extremely near.}$$

EXAMPLES:

1. Given $x^2 = 2$, or, which is the same thing, let the square root of 2 be found.

M

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Suppose the root, found by trial, to be 1.4; then we shall have $m=2$, $r=1.4$, $n=2$, and $v=\frac{2 \times 1.96}{2-1.96}=98$.

$$\text{And, therefore, } x=r+\frac{r \times 6v+n+1}{v \times 6v+4n-2}=1.4+\frac{1.4 \times 591}{98 \times 594}=1.4+\frac{197}{70 \times 198}=1.4+\frac{197}{13860}=1.41421356=\text{root required nearly.}$$

And, if the second approximation be used, the root will be found $=1.41421356236$, which is true to the last place of decimals.

2. Given $x^3=500$; or let it be required to extract the cube root of 500.

Suppose the root, found by trial, to be 8; then we shall have $m=500$, $r=8$, $n=3$, and $v=\frac{3 \times 512}{-12}=-128$;

$$\text{And, therefore } x=r+\frac{r \times 6v+n+1}{v \times 6v+4n-2}=8-\frac{.063}{7.93}=\text{for the 1st approximation.}$$

$$\text{Or } x=r+\frac{r \times 2v+n}{2v+2n-1 \times v+\frac{1}{6} \times n-1 \times 2n-1}=8-\frac{6072}{96389}=7.937005259936 \text{ for the 2d approximation,}$$

which is true to the last place of decimals.

3. Let it be required to find the cube root of 2.

Ans. 1.259921049894.

4. What is the cube root of 117? Ans. 4.89697.

5. What is the sursolid, or 5th root, of 125000? Ans. 10.456389.

6. It is required to find the 7th root of 100000.

Ans. 5.1794746792.

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PROBLEM IX.

To find the root of an exponential equation.

R U L E.*

1. Find, by trial, two numbers, as near the true root as possible, and substitute them in the given equation instead of the unknown quantity, marking the errors which arise from each of them.

2. Multiply the difference of the two numbers, found by trial, by the least error, and divide the product by the difference of the errors, when they are alike, and by their sum when they are unlike.

3. Add the quotient, last found, to the number belonging to the least error, when that number is too little, and subtract it when too great, and the result will give the true root *nearly*.

4. Take this root and the nearest of the former, and, by proceeding in like manner, a root will be had still nearer than before; and so on to any degree of exactness required.

E X A M P L E S :

1. Given $x^x = 100$, to find the value of x by approximation.

By the nature of logarithms $x \times \log. x = \log. 100 = 2$.

And, since x is found, by trial, to be greater than 3 and less than 4;

Let, therefore, 3.5 and 3.6 be the two supposed values of x .

* The rule for solving exponential equations was invented by M. Jean Bernoulli, and published in the *Leipsic Acts*, 1697.

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Then the $\log. x = \log. 3.5 = .5440680$; and $x \times$
 $\log. x = 1.9042380$

2

$-.0957620 = 1^{\text{st}} \text{ error, too little.}$

And the $\log. x = \log. 3.6 = .5563025$; and

$x \times \log. x = 2.0026890$

2

$.0026890 = 2^{\text{d}} \text{ error, too great.}$

1st number 3.5

1st error $-.095762$

2^d number 3.6

2^d error $+.002689$

$0.1 = \text{diff.}$

$.098451 = \text{sum.}$

$0.1 \times .002689 = .00273 = \text{correction,}$

$.098451$

2^d number 3.60000

correction $-.00273$

$3.59727 = x = \text{root nearly.}$

Again, suppose $x = 3.597$; then we shall have

$\log. x = .5559404$, and

$x \times \log. x = 1.9997176$, which subtracted from 2
 gives $.0002824$ the 3^d error, too little.

2^d number 3.600

2^d error $+.0026890$

3^d number 3.597

3^d error $-.0002824$

$.003 = \text{diff.}$

$.0029714 = \text{sum.}$

$.003 \times .0002824 = .000285 \text{ the correction.}$

$.0029714$

3^d number 3.597000

correction $+0.000285$

$3.597285 = x = \text{root required nearly.}$

OF INDETERMINATE, OR UNLIMITED PROBLEMS.

A problem is said to be *indeterminate* or *unlimited*, when the equations, expressing the conditions of a question, are less in number than the unknown quantities to be determined. And though such kind of problems are capable of innumerable answers, yet the results, in whole numbers, are generally limited to some determinate number, and may be obtained as follows.

P R O B L E M I.

To find the values of x and y , in the equation $ax = by + c$; where a , b , and c , are given numbers, which admit of no common divisor.

R U L E.

1. Let wb . stand for a *whole number*, and reduce the equation to $x = \frac{by+c}{a} = wb$.

2. Make $\frac{by+c}{a}$ to $\frac{dy+f}{a}$, by throwing all whole numbers out of it, till d and f be each less than a .

3. Subtract $\frac{dy+f}{a}$, or some multiple of it, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, or any other multiple of y , that comes.

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near the former, and the remainder will be a whole number.

4. Take this remainder, or any multiple of it, from some of the foregoing fractions, or from any whole number, which is nearly equal to it, and the remainder, in this case, will also be a whole number.

5. Proceed in the same manner with this last remainder; and so on till the co-efficient of y becomes equal to 1; or $\frac{y+g}{a} = wh. = p.$

6. Then will $y = ap - g$; where p may be any whole number whatever; and y being known, x may be found from the given equation.

EXAMPLES:

1. Given $19x = 14y - 11$, to find x and y in whole numbers.

$$\text{First, } x = \frac{14y - 11}{19} = wh.; \text{ and } \frac{19y}{19} = wh.$$

$$\text{Then, by subtraction, } \frac{19y}{19} - \frac{14y - 11}{19} = \frac{5y + 11}{19} = wh.$$

$$\text{Again, } \frac{5y + 11}{19} \times 4 = \frac{20y + 44}{19} = \frac{20y + 6}{19} + 2 = wh.$$

$$\text{or } \frac{20y + 6}{19} = wh.$$

$$\text{Therefore } \frac{20y + 6}{19} - \frac{19y}{19} = \frac{y + 6}{19} = wh. = p.$$

And $y = 19p - 6$; where, if p be taken $= 1$, for the least affirmative value of y , we shall have $y = 13$, and $x = 9$, the answer.

OF UNLIMITED PROBLEMS. 127

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

$$\text{Here, } x = \frac{8y-16}{3} = 2y-5 + \frac{2y-1}{3} = \text{wh. or. } \frac{2y-1}{3} = \text{wh.}$$

$$\text{And, } \frac{2y-1}{3} \times 2 = \frac{4y-2}{3} = \text{wh. But } \frac{3y}{3} \text{ is also } = \text{wh.}$$

$$\text{Therefore } \frac{4y-2}{3} - \frac{3y}{3} = \frac{y-2}{3} = \text{wh.} = p.$$

Whence $y=3p+2$; and, by taking $p=0$, we shall have $y=2$ and $x=0$, the answer.

3. Given $9x+13y=2000$; to find all the possible values of x and y in whole numbers.

$$\text{First, } x = \frac{2000-13y}{9} = 222-y + \frac{2-4y}{9} = \text{wh. or}$$

$$\frac{2-4y}{9} = \text{wh.}$$

$$\text{And, } \frac{2-4y}{9} \times 2 = \frac{4-8y}{9} = \text{wh. But, } \frac{9y}{9} \text{ is also } = \text{wh.}$$

$$\text{Therefore } \frac{9y}{9} + \frac{4-8y}{9} = \frac{y+4}{9} = \text{wh.} = p.$$

Whence $y=9p-4$; and, by taking $p=1$, we shall have $y=5$ and $x=215$.

And, by adding 9, continually, to the last value of y , and subtracting 13 from that of x , all the possible answers will stand as follows:

$$x = \{ 215. 189. 163. 137. 111. 85. 59. 33. 7$$

$$y = \{ 5. 23. 41. 59. 77. 95. 113. 131. 149.$$

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4. Given $24x=13y+16$, to find x and y in whole numbers.

Ans. $x=5$, and $y=8$.

5. Given $14x=4y+7$, to find x and y in whole numbers.

Ans. The question is impossible.

6. Given $27x=1600-16y$, to find x and y in whole numbers.

Ans. $y=19$, and $x=48$.

7. Given $87x+256y=15410$, to find the least value of x , and the greatest of y , in whole positive numbers.

Ans. $x=30$, and $y=12800$.

8. Given $5x+7y+11z=224$, to find all the possible values of x , y , and z , in whole numbers.

Ans. The number of answers is 60.

9. How many different ways is it possible to pay 100l. in guineas and pistoles only; a guinea being equal to 21s. and a pistole to 17s.

Ans. 6 different ways.

10. To determine whether it be possible to pay 100l. in guineas and moidores only.

Ans. The question is impossible.

11. A person bought 20 birds, of three different sorts, for 1s. 8d. the first sort at 4d. the second at $\frac{1}{2}$ d. and the third at $\frac{1}{4}$ a-piece; how many had he of each?

Ans. 3 of the 1st sort, 15 of the 2d, and 2 of the 3d.

12. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis d'ors; the question is, how must I acquit myself of the debt? the louis d'ors being valued at 17s.

Ans. I must pay him 13 guineas, and he must give me 16 louis d'ors.

13. To find in what year of Christ the cycle of the sun was 8, the cycle of the moon 10, and the cycle of indiction 10.

Ans. In the year 1567.

14. Twenty persons, consisting of men, women, and children, spend altogether 20s. the men spend 4s. a-piece; the women 6d. and the children 3d. how many were there of each sort?

Ans. 3 men, 15 women, and 2 children.

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15. A vintner has wine at 2s. 1s. 10d. and 1s. 6d. *per* gallon: how much of each sort must he take, so as to make a mixture of 30 gallons, to be sold at 1s. 8d. *per* gallon?

Ans. 16, 2, 12; 17, 4, 9; 18, 6, 6; or 19, 8, 3
of each sort.

16. To determine how many ways it is possible to pay 1000l. in crowns, guineas, and moidores, only.

Ans. 70734 different ways.

P R O B L E M II.

To find such a whole number x , as being divided by the given numbers a, b, c , &c. shall leave the given remainders f, g, h , &c.

R U L E.

1. Subtract each of the remainders from x , and divide the difference by a , and there will result $\frac{x-f}{a}, \frac{x-g}{a}, \frac{x-h}{a}$, &c. = whole numbers.

2. Call the value of x , in the first fraction, p , and substitute this quantity in the place of x in the second fraction.

3. Find the least value of p , in the second fraction, by the last problem, and call it r .

4. Let the value of x be found in terms of r , and substitute this quantity in the place of x in the third fraction.

5. Find the least value of r in the third fraction, and call it s ; and the value of x in terms of s , being substituted for x in the fourth fraction, and so on, will give the whole number required.

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EXAMPLES:

1. To find the least whole number, which, by being divided by 17, shall leave a remainder of 7; but being divided by 26, the remainder shall be 13.

Let x = number required.

Then $\frac{x-7}{17}$, and $\frac{x-13}{26}$ = whole numbers.

And, by putting $\frac{x-7}{17} = p$, we shall have $x = 17p + 7$;

Which value of x , being substituted in the 2d fraction,

gives $\frac{17p-6}{26} = \text{wh.}$ But $\frac{26p}{26}$ is also = wh.

And, therefore, $\frac{26p}{26} - \frac{17p-6}{26} = \frac{9p+6}{26} = \text{wh.}$

or $\frac{9p+6}{26} \times 3 = \frac{27p+18}{26} = p + \frac{p+18}{26} = \text{wh. or } \frac{p+18}{26} = \text{wh.} = r.$

Hence $p = 26r - 18$, and, by taking $r = 1$, we shall have $p = 8$.

And, consequently $x = 17 \times 8 + 7 = 143$, the number required.

2. To find a number which being divided by 11, 19, and 29, the remainders shall be 3, 5, and 10.

Let x = number required.

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$, and $\frac{x-10}{29}$ = whole numbers.

And, by putting $\frac{x-3}{11} = p$, we shall have $x = 11p + 3$;

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Which value of x , being substituted in the 2d fraction,

$$\text{gives } \frac{11p-2}{19} = \text{wh. or } \frac{11p-2}{19} \times 2 = \frac{22p-4}{19} = p +$$

$$\frac{3p-4}{19} = \text{wh.}$$

$$\text{Also, } \frac{3p-4}{19} = \text{wh. } \frac{3p-4}{19} \times 6 = \frac{18p-24}{19} = \frac{18p-5}{19}$$

$$1 = \text{wh. or } \frac{18p-5}{19} = \text{wh.}$$

$$\text{But } \frac{19p}{19} \text{ is, likewise, } = \text{wh. and, therefore, } \frac{19p}{19} =$$

$$\frac{18p-5}{19} = \frac{p+5}{19} = \text{wh.} = r.$$

$$\text{Hence, } p = 19r - 5, \text{ and } x = 19r - 5 \times 11 + 3 = 209r - 52.$$

And, by substituting this value of x in the 3d fraction,

$$\text{we shall have } \frac{209r-62}{29} = 7r - 2 + \frac{6r-4}{29} = \text{wh. or}$$

$$\frac{6r-4}{29} = \text{wh.}$$

$$\text{But, } \frac{6r-4}{29} \times 5 = \frac{30r-20}{29} = r + \frac{r-20}{29} = \text{wh. or}$$

$$\frac{r-20}{29} = \text{wh.} = s.$$

Therefore, $r = 29s + 20$; and, by putting $s = 0$, we shall have $r = 20$.

And, consequently, $x = 209 \times 20 - 52 = 4128 = \text{number required.}$

3. To find the least whole number, which being divided by 19, shall leave a remainder of 7; but being divided by 28, the remainder shall be 13.

Ans. 349.

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4. To find a number, which being divided by 3, 5, 7, and 2; will leave the remainders 2, 4, 6, and 0, respectively. *Ans.* 104.

5. To find the least whole number, which, being divided by 16, 17, 18, 19, and 20, shall leave 6, 7, 8, 9, and 10 remainders. *Ans.* 232550.

6. To find the least whole number, which, being divided by the nine digits, respectively, shall leave no remainders. *Ans.* 2520.

DIOPHANTINE PROBLEMS.

* *Diophantine problems* are those which relate to the finding of square and cube numbers, &c. and are such as are generally capable of a great variety of answers. They are so called from their inventor *Diophantus* of *Alexandria* in *Egypt*, who flourished in or about the third century, and is the first writer on Algebra we meet with amongst the ancients.

These questions are so exceedingly curious and abstruse, that nothing less than the most refined Algebra, applied with the utmost skill and judgment,

* That *Diophantus* was not the inventor of algebra, as has been generally imagined, is exceedingly obvious; for his method of applying it is such, as could only have been used in a very advanced state of the science. He no where speaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already sufficiently known; and seems to intend, not so much to teach it, as to cultivate and improve it, by solving such questions as, before this time, had been thought too difficult to be surmounted.

could ever surmount the difficulties which attend them. And, in this way, no man has ever extended the limits of the analytic art further than *Diophantus*, or discovered greater penetration and judgment in the application of it.

When we consider his work with attention, we are at a loss which to admire most, his wonderful sagacity, and peculiar artifice, in forming such positions as the nature of the problems required, or the more than ordinary subtilty of his reasoning upon them.

Every particular question puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which cannot but be very instructive to the mind, in conducting it through almost all difficulties of this kind, wherever they occur.

The following method of resolving these questions will be found of considerable service; but no general rule can be given, that will suit all cases; and therefore the solution must often be left to the sagacity and skill of the learner.

It is probable, therefore, that Algebra was known in the world, long before the time of *Diophantus*; but that the works of preceding writers have been destroyed by the ravages of time, or the depredations of ignorant barbarians.

His *Arithmetics*, out of which these problems were mostly collected, consisted originally of thirteen books; but the first six only are now extant. The best edition is said to be that published at *Paris*, by Monsieur *Bachet*, in the year 1621. And in this work, the subject is so skilfully handled, that the moderns, notwithstanding their other improvements, have been able to do little more than explain and illustrate his method.

Those who have succeeded best, in this respect, are *Vieta*, *Brancher*, *Kersey*, *De Billy*, *Ozanam*, *Prestet*, *Saunderson*, *Fermat*, and *Euler*.

R U L E.

1. For the root of the square or cube required, put one or more letters such, that when they are involved, either the given number, or the highest power of the unknown quantity, may vanish from the equation; and then if the unknown quantity be but of one dimension, the problem will be solved by reducing the equation.

2. But if the unknown quantity be still a square, or a higher power, some other new letters must be assumed to denote the root; with which proceed as before; and so on till the unknown quantity be but of one dimension; and from this all the rest will be determined.

E X A M P L E S :

1. * To divide a given square number (100) into two such parts, that each of them may be a square number.

Let $x^2 (= \square)$ be one of the parts, and then $100 - x^2$ will be the other part, which is also to be a square number.

* If $x - 10$ had been made the side of the second square, in this question, instead of $2x - 10$, the equation would have been $x^2 - 20x + 100 = 100 - x^2$; in which case, x , the side of the first square, would have been found $= 10$, and $x - 10$, or the side of the second square $= 0$; and for this reason the substitution, $x - 10$, was avoided; but $3x - 10$, $4x - 10$, or any other quantity of the same kind, would have succeeded equally as well as the former; though, in some cases, the results would have been less simple.

Assume the side of this second square $= 2x - 10$.

Then will $100 - x^2 = 2x - 10$ $^2 = 4x^2 - 40x + 100$;

And, by reduction, $x = 8$, and $2x - 10 = 6$.

Therefore 64 and 36 are the parts required.

THE SAME GENERALLY.

Let $a^2 =$ given square number, $x^2 (= \square) =$ one of its parts, and $a^2 - x^2 =$ the other, which is also to be a square number.

Assume the side of this second square $= rx - a$,

then will $a^2 - x^2 = rx - a$ $^2 = r^2 x^2 - 2arx + a^2$;

And, by reduction, $x = \frac{2ar}{r^2 + 1}$, and $rx - a = \frac{2ar^2}{r^2 + 1}$

$$-a = \frac{ar^2 - a}{r^2 + 1}.$$

Therefore $\left[\frac{2ar}{r^2 + 1} \right]^2$ and $\left[\frac{ar^2 - a}{r^2 + 1} \right]^2$ are the parts required; where a and r may be any numbers, taken at pleasure.

* 2. To divide a given number (13) consisting of two known square numbers (9 and 4) into two other square numbers.

If s and r be any two unequal numbers, of which s is the greater; then will $2rs$, $s^2 - r^2$ and $s^2 + r^2$, be the perpendicular, base, and hypotenuse of a right-angled triangle.

And from this canon two square numbers may be found, whose sum or difference shall be square number; for $2rs : + s^2 - r^2 = s^2 + r^2$, and $r^2 + s^2 - 2rs = s^2 - r^2$, or $s^2 + r^2 - s^2 - r^2 = 2rs$; and this when s and r are any numbers whatever.

* This question is considered, by *Diophantus*, as a very important one, being made the foundation, or *basis*, of most

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For the side of the first square sought, put $rx-3$; and for the side of the second, $sx-2$; r being the greater number, and s the less.

Then will $\overline{rx-3}^2 + \overline{sx-2}^2 = \overline{r^2x^2 - 6rx + 9} + \overline{s^2x^2 - 4sx + 4} = \overline{r^2 + s^2x^2 - 6r - 4sx + 13} = 13$, or $r^2 + s^2x^2 = 6r + 4sx$.

And, by reduction, $x = \frac{6r+4s}{r^2+s^2}$.

Whence $rx-3 = \frac{6r^2+4rs}{r^2+s^2} - 3 = \frac{3r^2+4rs-3s^2}{r^2+s^2} =$

side of the first square sought.

And, $sx-2 = \frac{6rs+4s^2}{r^2+s^2} - 2 = \frac{6rs-2r^2+2s^2}{r^2+s^2} =$ side

of the second.

So that if r be taken $= 2$, and $s = 1$, we shall have $\frac{3r^2+4rs-3s^2}{r^2+s^2} = \frac{17}{5}$ and $\frac{6rs-2r^2+2s^2}{r^2+s^2} = \frac{6}{5}$ for the sides of the squares, in numbers, as was required.

If $a^2 + b^2$ be put equal to the number to be divided, the general solution may be given in exactly the same manner.

of his other problems.—In the solution of it, given above, the values of r and s may be taken equal to any numbers whatever, provided the proportion of those numbers be not the same as that of 3 (a) to 2 (b), or $3+2$ ($a+b$) to $3-2$ ($a-b$). And the reason of this restriction is, that, if r and s were so taken, the sides of the squares sought, would come out the same as the sides of the known squares, which compose the given number, and therefore the operation would be useless.

The excellent old *Kersey*, after amplifying and illustrating this problem in a variety of ways, concludes his chapter thus: "For a further account of this rare speculation, see *Andersonus*, Theorem 2. of *Vieta's* mysterious doctrine of Angular Sections; and likewise *Herigonius*, at the latter end of the first Tome of his *Cursor Mathematicus*."

3. To find two square numbers, whose difference shall be equal to any given number (d).

Let d be resolved into any two unequal factors a and b ; a being the greater and b the less.

Also put x for the side of the lesser square sought, and $x+b$ = side of the greater.

Then $(x+b)^2 - x^2 = x^2 + 2bx + b^2 - x^2 = 2bx + b^2 = d(ab)$.

Or, dividing by b , $2x+b=a$.

Whence, $x = \frac{a-b}{2}$ = side of the lesser square sought,

and $x+b = \frac{a-b}{2} + b = \frac{a+b}{2}$ = side of the greater.

So that, by putting $d=60$, and $a \times b = 2 \times 30$, we shall have $\frac{30-2}{2} = 14$, and $\frac{30+2}{2} = 16$, or $14^2 = 196$, and $16^2 = 256$ for the squares in numbers; and so for any difference or factors whatever.

4. To find two numbers such, that if either of them be added to the square of the other, the sum shall be a square number.

Let the numbers sought be x and y , then $x^2 + y = \square$, and $y^2 + x = \square$.

And, if $r-x$ be assumed for the side of the first square $x^2 + y$, we shall have $x^2 + y = r^2 - 2rx + x^2$, or $y = r^2 - 2rx$.

Whence $2rx = r^2 - y$, or $x = \frac{r^2 - y}{2r}$.

Again, if $y+s$ be assumed for the side of the second square, we shall have $y^2 + x = \frac{r^2 - y}{2r} (y^2 + x) = y^2 + s^2 + 2sy + s^2$.

Whence $\frac{r^2 - y}{2r} = 2sy + s^2$, or $r^2 - y = 4rsy + 2rs^2$.

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And, consequently, $y = \frac{r^2 - 2rs^2}{4rs + 1}$ and $x = \frac{r^2 - y}{2r} = \frac{2r^2s + s^2}{4rs + 1}$.

So that $\frac{r^2 - 2rs^2}{4rs + 1}$ and $\frac{2r^2s + s^2}{4rs + 1}$ are the numbers required; where r and s may be taken at pleasure, provided that r be greater than $2s^2$.

5. To find two numbers, whose sum and difference shall be both square numbers.

Let x and $x^2 - x$ be the two numbers sought.

Then, since their sum is evidently a square number, one of the conditions of the questions will be answered.

There remains, therefore, only their difference $x^2 - 2x$ to be made a square.

And, if for the side of this square, there be put $x - r$, we shall have $x^2 - 2rx + r^2 = x^2 - 2x$, or $2rx - 2x = r^2$.

Whence, $x = \frac{r^2}{2r - 2}$ and $x^2 - x + \frac{r^2}{2r - 2} = \frac{r^2}{2r - 2}$.

So that $\frac{r^2}{2r - 2}$ and $\frac{r^2}{2r - 2} = \frac{r^2}{2r - 2}$ are the numbers required; where r may be taken at pleasure, provided it be greater than 1.

6. To find three numbers such, that not only the sum of all three of them, but also the sum of every two shall be a square number.

Let $4x$, $x^2 - 4x$ and $2x + 1$ be the three numbers sought.

Then $4x + x^2 - 4x (x^2)$, $x^2 - 4x + 2x + 1 (x^2 - 2x + 1)$, and $4x + x^2 - 4x + 2x + 1 (x^2 + 2x + 1)$, are evidently squares.

And, therefore, three of the conditions, mentioned in the question, are accomplished.

Whence, it remains only, to make the quantity $4x+2x+1$, or $6x+1$ = to a square.

Let, therefore, $6x+1=a^2$; and we shall have $x=\frac{a^2-1}{6}$.

And, consequently, $\frac{4a^2-4}{6}$, $\frac{a^2-1}{6}$, $\frac{4a^2-4}{6}$, and $\frac{2a^2-2}{6}+1$; or $\frac{2a^2-2}{3}$, $\frac{a^2-26a^2+25}{36}$, and $\frac{a^2+2}{3}$ are the numbers required; where a may be taken at pleasure, provided it be greater than 5.

7. To find three square numbers such, that the sum of every two of them shall be a square number*.

Let x^2 , y^2 , and z^2 be the numbers sought;
Then $x^2+z^2=\square$, $y^2+z^2=\square$, and $x^2+y^2=\square$.
Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, and $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, and $\frac{y}{z}=\frac{r^2-1}{2r}$,
we shall have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, and $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$ which are both evidently squares; and

* This question is capable of a great variety of answers; but the least roots, which have yet been found, in whole numbers, are 44, 117, and 240. See *Eléments d'Algebre*, par M. Euler, tome II. page 327.

therefore it remains only to make $\frac{x^2}{z^2} + \frac{y^2}{z^2} = \text{square number.}$

$$\text{But } \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{s^2-1}{2s} + \frac{r^2-1}{2r} = \frac{s^2-1}{4s^2} + \frac{r^2-1}{4r^2} \\ = \frac{4r^2 \times s^2 - 1}{4r^2 s^2} + \frac{4s^2 \times r^2 - 1}{4s^2 r^2} = \square;$$

$$\text{Or } r^2 \times s^2 - 1 + s^2 \times r^2 - 1 = r^2 \times s + 1 + s - 1 + s^2 \times r + 1 + r - 1 = \square.$$

And, by making $r-1=s+1$, or $r=s+2$, we shall have $s+2)^2 \times s+1)^2 \times s-1)^2 + s^2 \times s+3)^2 \times s+1)^2 = \square;$

$$\text{Or } s+2)^2 \times s-1)^2 + s^2 \times s+3)^2 = 2s^4 + 8s^3 + 6s^2 - 4s + 4 = \square.$$

Now, let the root of this square be assumed $= \frac{5}{4}s^2 - s + 2$,

$$\text{Then, } 2s^4 + 8s^3 + 6s^2 - 4s + 4 = \left(\frac{5}{4}s^2 - s + 2\right)^2 = \frac{25}{16}s^4 - \frac{5}{2}s^3 + 5s^2 + s^2 - 4s + 4; \text{ or } 2s^4 + 8s^3 = \frac{25}{16}s^4 - \frac{5}{2}s^3; \text{ or } 2s + 8 = \frac{25}{16}s - \frac{5}{2}.$$

$$\text{Whence } s = -24, \text{ and } r = -22.$$

$$\text{And, consequently, } \frac{x}{z} = \frac{s^2-1}{2s} = -\frac{575}{48}, \text{ and } \frac{y}{z} = \frac{r^2-1}{2r} = -\frac{483}{44};$$

$$\text{or } x = -\frac{575z}{48}, \text{ and } y = -\frac{483z}{44}.$$

In order, therefore, to have the answer in whole numbers, let $z=528$, and we shall have $x=6325$, and $y=5796$, or 528, 5796 and 6325 for the roots of the squares required.

8. To find a number x such, that $x+1$ and $x-1$ shall be both square numbers. *Ans.* $x = \frac{5}{4}$.

9. To find a number x such, that $x+128$ and $x+192$ shall be both squares. *Ans.* $x=97$.

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10. To find a number x such, that $x^2 + x$ and $x^2 - x$ may be both squares.

$$\text{Ans. } x = \frac{25}{24}.$$

11. To find two numbers x and y such, that $x + y$, $x^2 + y$ and $y^2 + x$ may be all squares.

$$\text{Ans. } x = \frac{1}{6} \text{ and } y = \frac{1}{12}.$$

12. To find three numbers in arithmetical progression such, that the sum of every two of them may be a square number.

$$\text{Ans. } 120\frac{1}{2}, 840\frac{1}{2}, \text{ and } 1560\frac{1}{2}.$$

13. To find three numbers such, that if to the square of every one of them the sum of the other two be added, the three sums shall be all squares.

$$\text{Ans. } \frac{8}{3}, \frac{16}{3} \text{ and } 1.$$

14. To find two numbers in proportion as 8 is to 15, and such that the sum of their squares shall make a square number.

$$\text{Ans. } 576 \text{ and } 1080.$$

15. To find three square numbers in arithmetical progression.

$$\text{Ans. } 1, 25, \text{ and } 49.$$

16. To find three square numbers in harmonical proportion.

$$\text{Ans. } 1225, 49, \text{ and } 25.$$

17. To find four numbers such, that if a square number (100) be added to the product of every two of them, the sums shall be all squares.

$$\text{Ans. } 2, 32, 88 \text{ and } 168.$$

18. To find two numbers such, that their difference may be equal to the difference of their squares, and that the sum of their squares shall be a square number.

$$\text{Ans. } \frac{4}{7} \text{ and } \frac{3}{7}.$$

19. To find three numbers in geometrical proportion such, that every one of them being increased by a given number 19, shall make square numbers.

$$\text{Ans. } 81, \frac{5}{4} \text{ and } \frac{25}{1296}.$$

20. To find two numbers such, that if their product be added to the sum of their squares, it shall make a square number.

$$\text{Ans. } 5 \text{ and } 3, 8 \text{ and } 7, 16 \text{ and } 5, \text{ \&c.}$$

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21. To divide a given number (10) into four such parts, that the sum of every three of them may make a square number.

$$\text{Ans. } 1, 6, \frac{186}{289} \text{ and } \frac{681}{289}.$$

22. To find three square numbers such, that their sum being severally added to their three sides shall make square numbers.

$$\text{Ans. } \frac{4418}{62920}, \frac{13254}{62920}, \text{ and } \frac{19881}{62920} = \text{roots required.}$$

23. To find two numbers such, that their sum being increased and lessened, either by their difference, or the difference of their squares, the sums and remainders shall be all squares.

$$\text{Ans. } \frac{49}{36} \text{ and } \frac{1}{36}.$$

24. To find two numbers such, that not only each number, but also their sum and their difference, being increased by unity, shall be all square numbers.

$$\text{Ans. } 3024 \text{ and } 5624.$$

25. To find three numbers such, that whether their sum be added to, or subtracted from the square of every particular number, the numbers thence arising shall be all squares.

$$\text{Ans. } \frac{406}{96}, \frac{518}{96} \text{ and } \frac{791}{96}.$$

26. To find three square numbers such, that the sum of their squares shall also be a square number.

$$\text{Ans. } 9, 16, \text{ and } \frac{144}{25}.$$

27. To find three square numbers such, that the difference of every two of them shall be a square number.

$$\text{Ans. } 485809, 34225 \text{ and } 23409.$$

28. To divide any given cube number (8) into three other cube numbers.

$$\text{Ans. } \frac{64}{27}, \frac{125}{27} \text{ and } 1.$$

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29. Two cube numbers (8 and 1) being given, to find two other cube numbers, whose difference shall be equal to the sum of the given cubes.

$$\text{Ans. } \frac{8000}{343}, \text{ and } \frac{4913}{343}.$$

30. To divide a given number (28) composed of two cube numbers (27 and 1) into two other cube numbers.

$$\text{Ans. } \frac{63284705}{21446828} \text{ and } \frac{28340511}{21446828} \text{ the roots.}$$

31. To find three cube numbers such, that if from every one of them a given number (1) be subtracted, the sum of the remainders shall be a square.

$$\text{Ans. } \frac{4913}{3375}, \frac{21952}{3375} \text{ and } 8.$$

32. To find three numbers such, that if they be severally added to the cube of their sum, the three sums thence arising shall be all cubes.

$$\text{Ans. } \frac{1538}{157464}, \frac{18577}{157464} \text{ and } \frac{23625}{157464}.$$

33. To find three numbers in arithmetical proportion such, that the sum of their cubes shall be a cube.

$$\text{Ans. } 3, 4, 5, \text{ or } 149, 256, \text{ and } 363, \&c.$$

34. To find three cube numbers such, that their sum shall be a cube number.

$$\text{Ans. } 3^3, 4^3, \text{ and } 5^3, \text{ or } 21^3, 19^3, 18^3, \&c.$$

Of the SUMMATION, and INTERPOLATION, of INFINITE CONVERGING SERIES.

The doctrine of infinite series is a subject which has engaged the attention of the greatest Mathematicians in all ages; and is, perhaps, one of the most abstruse and difficult branches of abstract mathematics.

To find the sum of a series, the number of whose terms is inexhaustible, or infinite, has been considered by some as an extravagant paradox, or a thing impossible to be done. But this difficulty will be easily removed, by considering, that every finite magnitude whatever is divisible *in infinitum*, or consists of an infinite number of parts, whose aggregate, or sum, is equal to the quantity first proposed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number which we can possibly conceive, or of which we have any notion, must always be determinate and finite; so that a greater may be still assigned, and a greater after this; and so on, without a possibility of ever coming to an end of the increase or addition.

And this inexhaustibility, in the nature of numbers, is, therefore, all that we can distinctly comprehend by their infinity; for though we can easily conceive that a finite quantity may become greater and greater without end, yet we are not from thence enabled to form any notion of the *ultimatum*, or last magnitude, which is incapable of further augmentation.

We cannot, therefore, apply to an infinite series the common notion of a sum, or a collection of several particular numbers, that are joined and added together, one after another; for this supposes, that those particulars are all known and determined. But as every series generally observes some particular law, and continually approaches towards a term or limit, we can easily conceive it to be a whole, of its own kind, and that it must have a certain real value, whether that value be determinable or not.

Thus, in many series, a number is assignable, beyond which, no number of its terms can ever

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reach, or indeed ever be equal to it; but yet may approach to it, in such a manner, as to want less than any given difference. And this we may call the value, or sum of the series; not as being a number found by the common method of addition, but as such a limitation of the value of the series, taken in all its infinite capacity, that if it were possible to add them all together, one after another, the sum would be equal to that number.

Again, in other series, the value has no limitation; and this may be expressed by saying, that the sum of the series is infinitely great; or, which is the same thing, that it has no determinate and assignable value; but may be carried on, to such a length, as that its sum shall exceed any given number whatever.

According to the common rule for summing up a finite progression of a geometric decreasing series, where r is the ratio, l the first term, and a the least, the sum is $\frac{rl-a}{r-1}$. And if we suppose a , the less extreme, to be actually decreased to 0, then the sum of the whole series will be $\frac{rl}{r-1}$. For it is demonstrable, that the sum of no assignable number of terms of the series can ever be equal to that quotient; and yet no number less than it, will ever be equal to the value of the series.

Whatever consequences, therefore, follow from the supposition of $\frac{rl}{r-1}$ being the true and adequate value of the series, taken in all its infinite capacity, as if all the parts were actually determined and added together, they can never be the occasion of any assignable error, in any operation or demonstration,

where it is used in that sense; because if you say that it exceeds that value, it is demonstrable that this excess must be less than any assignable difference, which is, in effect, no difference at all; and therefore, the supposed error will likewise be no error, and consequently $\frac{rl}{r-1}$ may be looked upon as expressing the adequate and just value of the infinite series.

But we are further satisfied of the reasonableness of this doctrine, by finding, in fact, that a finite quantity does actually convert into an infinite series, as appears in the case of circulating decimals. Thus, $\frac{2}{3}$ turned into a decimal is $=.6666$, &c. $=\frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000}$, &c. continued *ad infinitum*. But this is plainly a geometric series, from $\frac{6}{10}$, in the continued ratio of 10 to 1, and the sum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from whence it was originally derived. And the same may be shewn of many other series, and of all circulating decimals in general.

P R O B L E M I.

Any series being given to find the several orders of differences.

R U L E.

1. Take the first term from the second, the second from the third, the third from the fourth, &c. and the remainders will form a new series, called the *first order of differences*.

2. Take the first term of this last series from the second, the second from the third, the third from

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the fourth, &c. and the remainders will form another new series, called the *second order of differences*.

3. Proceed, in like manner, for the *third, fourth, fifth, &c.* orders of differences; and so on till they terminate, or are carried as far as is thought necessary.

EXAMPLES:

1. To find the several orders of differences in the series 1, 4, 9, 16, 25, 36, &c.

1, 4, 9, 16, 25, 36, &c.

3, 5, 7, 9, 11, &c. 1st diff.

2, 2, 2, 2, &c. 2d diff.

0, 0, 0, &c.

2. To find the several orders of differences in the series 1, 8, 27, 64, 125, 216, &c.

1, 8, 27, 64, 125, 216, &c.

7, 19, 37, 61, 91, &c. 1st diff.

12, 18, 24, 30, &c. 2d diff.

6, 6, 6, &c. 3d diff.

0, 0, &c.

3. To find the several orders of differences in the series 1, 3, 6, 10, 15, 21, &c.

Ans. 1st 2, 3, 4, 5, &c. 2d 1, 1, 1, &c.

4. To find the several orders of differences in the series 1, 6, 20, 50, 105, 196, &c.

Ans. 1st 5, 14, 30, 55, 91, &c. 2d 9, 16, 25, 36, &c. 3d 7, 9, 11, &c. 4th 2, 2, &c.

PROBLEM II.

Any series a, b, c, d, e , &c. being given, to find the first term of the n th. order of differences.

R U L E.

Let δ stand for the first term of the n th. differences.

Then will $a - nb + n \times \frac{n-1}{2}c - n \times \frac{n-1}{2} \times \frac{n-2}{3}d + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}e$, &c. to $n+1$ terms $= \delta$, when n is an even number.

And $-a + nb - n \times \frac{n-1}{2}c + n \times \frac{n-1}{2} \times \frac{n-2}{3}d - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}e$, &c. to $n+1$ terms $= \delta$, when n is an odd number.

E X A M P L E S :

1. Required the first term of the third order of differences, of the series 1, 5, 15, 35, 70, &c.

Let a, b, c, d, e , &c. $= 1, 5, 15, 35, 70$, &c. and $n=3$.

Then $-a + nb - n \times \frac{n-1}{2}c + n \times \frac{n-1}{2} \times \frac{n-2}{3}d = -a + 3b - 3c + d = -1 + 15 - 45 + 35 = 4 = \text{the first term required.}$

2. Required the first term of the fourth order of differences of the series, 1, 8, 27, 64, 125, &c.

Let a, b, c, d, e , &c. $= 1, 8, 27, 64, 125$, &c. and $n=4$.

Then $a - nb + n \times \frac{n-1}{2}c - n \times \frac{n-1}{2} \times \frac{n-2}{3}d + n \times$

$$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e, = a - 4b + 6c - 4d + e = 1 - 32 + 162 - 256 + 125 = 0; \text{ so that the first term of the fourth order is 0.}$$

3. Required the first term of the fifth order of differences, of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \&c.$

Ans. $-\frac{1}{32}.$

4. Required the first term of the 8th order of differences of the series, $1, 3, 9, 27, 81, \&c.$ *Ans.* $256.$

PROBLEM III.

To find the n th. term of the series, $a, b, c, d, e, \&c.$

RULE.

Let $d^I, d^{II}, d^{III}, d^{IV}, \&c.$ be the first of the several orders of differences, found as in the last problem:

$$\text{Then will } a + \frac{n-1}{1} d^I + \frac{n-1}{1} \times \frac{n-2}{2} d^{II} + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} d^{III} + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} d^{IV}, \&c. \text{ be } = n\text{th. term required.}$$

EXAMPLES:

1. To find the 12th term of the series $2, 6, 12, 20, 30, \&c.$

$2, 6, 12, 20, 30, \&c.$

$4, 6, 8, 10, \&c.$

$2, 2, 2, \&c.$

$0, 0, \&c.$

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Here 4 and 2 are the first terms of the differences ;

Let, therefore, $4 = d^1$, $2 = d^{11}$, and $n = 12$.

Then $a + \frac{n-1}{1}d^1 + \frac{n-1}{1} \times \frac{n-2}{2}d^{11} = 2 + 11d^1 + 55d^{11} = 2 + 44 + 110 = 156 = 12^{th}$ term required.

2. Required the 20th term of the series 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21, &c.

2, 3, 4, 5, 6, &c.

1, 1, 1, 1, &c.

0, 0, 0, &c.

Here 2 and 1 are the first terms of the differences.

Let, therefore, $2 = d^1$, $1 = d^{11}$, and $n = 20$.

Then $a + \frac{n-1}{1}d^1 + \frac{n-1}{1} \times \frac{n-2}{2}d^{11} = 1 + 19d^1 + 171d^{11} = 1 + 38 + 171 = 210 = 20^{th}$ term required.

3. Required the 15th term of the series 1, 4, 9, 16, 25, 36, &c. *Ans.* 225.

4. Required the 20th term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

P R O B L E M IV.

To find the sum of n terms of the series $a, b, c, d, e, \&c.$

R U L E.

Let $d^1, d^{11}, d^{111}, d^{1111}, \&c.$ be the first of the several orders of differences.

Then will $na + n \times \frac{n-1}{2}d^1 + n \times \frac{n-1}{2} \times \frac{n-2}{3}d^{11} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}d^{111} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}d^{1111} + \&c.$

$\frac{n-3}{4} \times \frac{n-4}{5} d^{1v}$, &c. = to the sum of n terms of the series.

EXAMPLES:

1. To find the sum of n terms of the series 1, 2, 3, 4, 5, 6, &c.

1, 2, 3, 4, 5, 6, &c.

1, 1, 1, 1, 1, &c.

0, 0, 0, 0, 0, &c.

Here 1 and 0 are the first terms of the differences.

Let, therefore, $a=1$, $d^1=1$, and $d^{11}=0$;

Then will $na + n \times \frac{n-1}{2} d^1 = n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$
sum of n terms, as required.

2. To find the sum of n terms of the series $1^2, 2^2, 3^2, 4^2, 5^2$, &c. or 1, 4, 9, 16, 25, &c.

1, 4, 9, 16, 25, &c.

3, 5, 7, 9, &c.

2, 2, 2, &c.

0, 0, &c.

Here 3 and 2 are the first terms of the differences:

Let, therefore, $a=1$, $d^1=3$ and $d^{11}=2$.

Then will $na + n \times \frac{n-1}{2} d^1 + n \times \frac{n-1}{2} \times \frac{n-2}{3} d^{11} = n +$
 $3n \times \frac{n-1}{2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{3n^2-n}{2} + \frac{n^3-3n^2+2n}{3}$
 $= \frac{n \times n + 1 \times 2n + 1}{6} =$ sum of n terms, as required.

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* 3. To find the sum of n terms of the series $1^3, 2^3, 3^3, 4^3, 5^3, \&c.$ or $1, 8, 27, 64, 125, \&c.$

1, 8, 27, 64, 125, &c.

7, 19, 37, 61, &c.

12, 18, 24, &c.

6, -6, &c.

0, &c.

Here the first terms of the differences are 7, 12 and 6.
Let, therefore, $a=1, d^1=7, d^{11}=12$, and $d^{111}=6$.

$$\begin{aligned} \text{Then will } na + n \times \frac{n-1}{2} d^1 + n \times \frac{n-1}{2} \times \frac{n-2}{3} d^{11} \\ + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} d^{111} = n + 7n \times \frac{n-1}{2} + 12n \\ \times \frac{n-1}{2} \times \frac{n-2}{3} + 6n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = \frac{7n^2-5n}{2} \end{aligned}$$

* The sums of a series of powers of the natural numbers 1, 2, 3, 4, 5, &c. may be exhibited as follows :

1	$1 + 2 + 3 + 4, \&c. + n = \frac{n^2+n}{2}$
2	$1^2 + 2^2 + 3^2 + 4^2, \&c. + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
3	$1^3 + 2^3 + 3^3 + 4^3, \&c. + n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$
4	$1^4 + 2^4 + 3^4 + 4^4, \&c. + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$
5	$1^5 + 2^5 + 3^5 + 4^5, \&c. + n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$
6	$1^6 + 2^6 + 3^6 + 4^6, \&c. + n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$
7	$1^r + 2^r + 3^r + 4^r, \&c. + n^r = \frac{n^{r+1}}{r+1} + \frac{n^r}{2} + \frac{rn^{r-1}}{3 \cdot 4}$

$$\frac{r \cdot r - 1 \cdot r - 2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} n^{r-3} + \frac{r \cdot r - 1 \cdot r - 2 \cdot r - 3 \cdot r - 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 6} n^{r-5} - \&c.$$

$$+ 2n^3 - 6n^2 + 4n + \frac{n^4 - 6n^3 + 11n^2 - 6n}{4} = \frac{n^4 + 2n^3 + n^2}{4}$$

= sum of n terms, as required.

4. To find the sum of n terms of the series 2, 6, 12, 20, 30, &c.

$$\text{Ans. } \frac{n \times n + 1 \times n + 2}{3}.$$

5. To find the sum of n terms of the series 1, 3, 6, 10, 15, &c.

$$\text{Ans. } \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}.$$

6. To find the sum of n terms of the series 1, 4, 10, 20, 35, &c.

$$\text{Ans. } \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}.$$

7. To find the sum of n terms of the series $1^4, 2^4, 3^4, 4^4, \&c.$ or 1, 16, 81, 256, &c.

$$\text{Ans. } \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

The sums of n terms of a series of triangular numbers may also be exhibited as follows:

1	$1 + 1 + 1 + 1, \&c. = n$	$\frac{n \times 1}{1}$
2	$1 + 2 + 3 + 4, \&c. =$	$\frac{n \cdot n + 1}{1 \cdot 2}$
3	$1 + 3 + 6 + 10, \&c. =$	$\frac{n \cdot n + 1 \cdot n + 2}{1 \cdot 2 \cdot 3}$
4	$1 + 4 + 10 + 20, \&c. =$	$\frac{n \cdot n + 1 \cdot n + 2 \cdot n + 3}{1 \cdot 2 \cdot 3 \cdot 4}$
5	$1 + 5 + 15 + 35, \&c. =$	$\frac{n \cdot n + 1 \cdot n + 2 \cdot n + 3 \cdot n + 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$
6	$1 + 6 + 21 + 56, \&c. =$	$\frac{n \cdot n + 1 \cdot n + 2 \cdot n + 3 \cdot n + 4 \cdot n + 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$
	&c.	&c.

PROBLEM V.

The series a, b, c, d, e , &c. being given, whose terms are at an unit's distance from each other; to find any intermediate term by interpolation.

RULE.

Let x be the distance of any term y to be interpolated, and d^1, d^{11}, d^{111} , &c. the first terms of the differences:

$$\text{Then will } a + xd^1 + x \times \frac{x-1}{2} d^{11} + x \times \frac{x-1}{2} \times \frac{x-2}{3} d^{111} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} d^{1111} \text{ \&c. } = y.$$

EXAMPLES:

1. Given the logarithmic sines of $1^\circ 0'$, $1^\circ 1'$, $1^\circ 2'$, and $1^\circ 3'$ to find the sine of $1^\circ 1' 40''$.

$1^\circ 0'$	$1^\circ 1'$	$1^\circ 2'$	$1^\circ 3'$
Sines 8.2418553	8.2490332	8.2560943	8.2630424
	71779	70611	69481
		-1168	-1130
			38

Here the first terms of the differences are 71779, -1168, and 38.

Let, therefore, $x = 1^\circ 1' 40'' - 1^\circ 0' = 1' 40'' = 1\frac{2}{3}$ = distance of y , the term to be interpolated; and $d^1 = 71779$, $d^{11} = -1168$, and $d^{111} = 38$.

$$\text{Then will } y = a + xd^1 + x \times \frac{x-1}{2} d^{11} + x \times \frac{x-1}{2} \times \frac{x-2}{3} d^{111} = a + \frac{5}{3} d^1 + \frac{5}{9} d^{11} - \frac{5}{81} d^{111} = 8.2418553 +$$

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.0119631—.0000694—.0000002=8.2537533=*fine*
of $1^{\circ} 1' 40''$, as was required.

2. Given the series $\frac{1}{3^0}, \frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}$, &c. to find the term which stands in the middle between $\frac{1}{3^2}$ and $\frac{1}{3^3}$.
Ans. $\frac{1}{3^2.5}$.

3. Given the natural tangents of $88^{\circ}, 54', 88^{\circ}, 55', 88^{\circ}, 56', 88^{\circ}, 57', 88^{\circ}, 58'$, and $88^{\circ}, 59'$; to find the tangent of $88^{\circ}, 58', 18''$.

Ans. 55.711144.

PROBLEM VI.

Having given a series of equidistant terms a, b, c, d, e, &c. whose first differences are small; to find any intermediate term by interpolation.

RULE.

Find the value of the unknown quantity in the equation which stands against the given number of terms, in the following table, and it will give the term required.

1	$a - b = 0$
2	$a - 2b + c = 0$
3	$a - 3b + 3c - d = 0$
4	$a - 4b + 6c - 4d + e = 0$
5	$a - 5b + 10c - 10d + 5e - f = 0$
6	$a - 6b + 15c - 20d + 15e - 6f + g = 0$
7	$a - nb + n \times \frac{n-1}{2} c - n \times \frac{n-1}{2} \times \frac{n-2}{3} d \&c. = 0.$

EXAMPLES:

1. Given the logarithms of 101, 102, 104, and 105; to find the logarithm of 103.

Here the number of terms are 4,

Therefore against 4, in the table, we have $a - 4b +$

$$6c - 4d + e = 0; \text{ or } c = \frac{4 \times b + d - a + e}{6}.$$

$$\text{Whence } \begin{cases} a = 2.0043214 \\ b = 2.0086002 \\ d = 2.0170333 \\ e = 2.0211893 \end{cases}$$

$$4 \times b + d = 16.1025340$$

$$a + e = 4.0255107$$

$$6) 12.0770233$$

$$2.0128372 = \log. \text{ of } 103, \text{ as required.}$$

2. Given the cube roots of 45, 46, 47, 48, and 49; to find the cube root of 50. *Ans.* 3.684033.

3. Given the logarithms of 50, 51, 52, 54, 55, and 56; to find the logarithm of 53.

$$\text{Ans. } 1.7242758695.$$

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the sum (S) of n terms of the series 1, 2, 3, 4, 5, 6, &c.

$$\text{First } 1 + 2 + 3 + 4 + 5, \text{ \&c. } \dots n = S.$$

$$\text{And } n + n - 1 + n - 2 + n - 3 + n - 4, \text{ \&c. } \dots 1 = S.$$

$$\text{Therefore } n + 1 + n + 1 + n + 1 + n + 1, \text{ \&c. } \dots n + 1 = 2S.$$

$$\text{And consequently } n + 1 \times n = 2S; \text{ or } S = \frac{n^2 + n}{2} =$$

sum required.

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To find the sum (S) of n terms of the series 1, 3, 5, 7, 9, 11, &c.

First $1+3+5+7+9$, &c. . . . $2n-1=S$.

And $2n-1+2n-3+2n-5+2n-7+2n-9$, &c. . . . $1=S$.

Therefore $2n+2n+2n+2n+2n$, &c. . . . $2n=2S$.

And, consequently, $2n \times n = 2S$; or $S = \frac{2n \times n}{2} = n^2 =$
sum required.

3. Required the sum (S) of n terms of the series $a+a+d+a+2d+a+3d+a+4d$, &c.

First $a+a+d+a+2d+a+3d$, &c. . . . $a+n-1 \times d=S$.

And $a+nd-d+a+nd-2d+a+nd-3d+a+nd-4d$, &c. . . . $a=S$.

Therefore $2a+nd-d+2a+nd-d+2a+nd-d$, &c. $2a+nd-d=2S$.

And, consequently, $2a+nd-d \times n = 2S$; or $S = \frac{2a+nd-d \times n}{2} =$ sum required.

OR THUS:

First, $a+a+d+a+2d+a+3d+a+4d$, &c.

$$= \left\{ \begin{array}{l} 1+1+1+1+1, \text{ &c. } \times a \\ +0+1+2+3+4, \text{ &c. } \times d \end{array} \right\} = S.$$

But n terms of $1+1+1+1+1$, &c. $= n$.

$$\text{Ditto of } 0+1+2+3+4, \text{ &c. } = \frac{n \times n-1}{2}$$

$$\text{And therefore } S = na + \frac{n \times n-1 \times d}{2} = 2a+nd-d \times \frac{n}{2}$$

$\frac{n}{2}$ as before.

4. To find the sum (S) of n terms of the series 1, x , x^2 , x^3 , x^4 , &c.

First, $1 + x + x^2 + x^3 + x^4$, &c. $\dots x^{n-1} = S$.

And $x + x^2 + x^3 + x^4 + x^5$, &c. $\dots x^n = Sx$.

Therefore $-1 + x^n = Sx - S$;

Or $S = \frac{x^n - 1}{x - 1} = \text{sum required.}$

And, when x is a proper fraction, the sum of the series, continued *ad infinitum*, may be found in the same manner.

Thus, $1 + x + x^2 + x^3 + x^4$, &c. $= S$.

And $x + x^2 + x^3 + x^4 + x^5$, &c. $= Sx$.

Therefore $-1 = Sx - S$; or $S - Sx = 1$.

Whence $S = \frac{1}{1-x} = \text{sum of an infinite number of terms, as required.}$

5. Required the sum (S) of the circulating decimal .999999, &c. continued *ad infinitum*.

First, .999999, &c. $= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}$, &c.

$= 9 \times \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} \right)$, &c. $= S$.

Or, $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000}$, &c. $= \frac{S}{9}$.

Therefore, $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= \frac{10S}{9}$.

And, $1 = \frac{10S}{9} - \frac{S}{9} = \frac{9S}{9}$.

Whence $S = 1 = \text{sum required.}$

6. Required the sum (S) of the series $a^2 + \overline{a+d}^2 + \overline{a+2d}^2 + \overline{a+3d}^2 + \overline{a+4d}^2$, &c. continued to n terms.

First, $a^2 = a^2$

$$a+d)^2 = a^2 + 2 \times 1ad + 1d^2$$

$$a+2d)^2 = a^2 + 2 \times 2ad + 4d^2$$

$$a+3d)^2 = a^2 + 2 \times 3ad + 9d^2$$

$$a+4d)^2 = a^2 + 2 \times 4ad + 16d^2$$

&c.

&c.

$$\text{Therefore, } S = \begin{cases} \text{Sum of } n \text{ terms of } 1+1+1+1, \text{ \&c.} \\ \times a^2 \\ + \dots \text{ ditto of } 0+1+2+3, \text{ \&c.} \\ \times 2ad \\ + \dots \text{ ditto of } 0+1+4+9, \text{ \&c.} \\ \times d^2 \end{cases}$$

But n terms of $1+1+1+1, \text{ \&c.} = n$

$$\text{Ditto of } 0+1+2+3, \text{ \&c.} = \frac{n \times n-1}{1 \times 2}$$

$$\text{And ditto of } 0+1+4+9, \text{ \&c.} = \frac{n \times n-1 \times 2n-1}{1 \times 2 \times 3}$$

$$\text{Whence } S = n \times a^2 + n \times n-1 \times ad + \frac{n \times n-1 \times 2n-1}{1 \times 2 \times 3}$$

$$\times d^2 = na^2 + nad \times n-1 + \frac{nd^2 \times n-1 \times 2n-1}{1 \times 2 \times 3} = \text{sum}$$

required.

7. Required the sum (S) of the series $a^3 + a+d)^3 + a+2d)^3 + a+3d)^3 + a+4d)^3, \text{ \&c.}$ continued to n terms.

First, $a^3 = a^3$

$$a+d)^3 = a^3 + 3 \times 1a^2d + 3 \times 1ad^2 + 1d^3$$

$$a+2d)^3 = a^3 + 3 \times 2a^2d + 3 \times 4ad^2 + 8d^3$$

$$a+3d)^3 = a^3 + 3 \times 3a^2d + 3 \times 9ad^2 + 27d^3$$

$$a+4d)^3 = a^3 + 3 \times 4a^2d + 3 \times 16ad^2 + 64d^3$$

P2

$$\text{Therefore, } S = \left\{ \begin{array}{l} \text{Sum of } n \text{ terms of } \overline{1+1+1+1}, \text{ \&c.} \\ \times a^3 \\ + \dots \text{ ditto of } \overline{0+1+2+3}, \text{ \&c.} \\ \times 3a^2d \\ + \dots \text{ ditto of } \overline{0+1+4+9}, \text{ \&c.} \\ \times 3ad^2 \\ + \dots \text{ ditto of } \overline{0+1+8+27}, \text{ \&c.} \\ \times d^3 \end{array} \right.$$

But n terms of $1+1+1+1$, &c. $= n$.

Ditto ... of $0+1+2+3$, &c. $= \frac{n \times n - 1}{1 \times 2}$

Ditto ... of $0+1+4+9$, &c. $= \frac{n \times n - 1 \times 2n - 1}{1 \times 2 \times 3}$

Ditto ... of $0+1+8+27$, &c. $= \frac{n^4 - 2n^3 + n^2}{2 \times 2}$

Whence $S = na^3 + \frac{n \times n - 1 \times 3a^2d}{1 \times 2} +$

$\frac{n \times n - 1 \times 2n - 1 \times 3ad^2}{1 \times 2 \times 3} + \frac{n^4 - 2n^3 + n^2 \times d^3}{2 \times 2} = \text{sum}$
required.

8. Required the sum (S) of n terms of the series $1+3+7+15+31$, &c.

The terms of this series are evidently equal to 1 , $1+2$, $1+2+4$, $1+2+4+8$, &c. or the successive sums of the geometrical progression $1, 2, 4, 8, 16$, &c.

Let, therefore, $a=1$, and $r=2$, and we shall have $a+ar+ar^2+ar^3+ar^4$, &c. $= 1+2+4+8+16$, &c.

But the sums of $1, 2, 3, 4$, &c. terms of this series, are

$$\begin{array}{l|l}
 1 & \frac{ar-a}{r-1} = \overline{r-1} \times \frac{a}{r-1} \\
 2 & \frac{ar^2-a}{r-1} = \overline{r^2-1} \times \frac{a}{r-1} \\
 3 & \frac{ar^3-a}{r-1} = \overline{r^3-1} \times \frac{a}{r-1} \\
 4 & \frac{ar^4-a}{r-1} = \overline{r^4-1} \times \frac{a}{r-1} \\
 \hline
 & \text{\textcircled{C}c.} \quad \text{\textcircled{C}c.}
 \end{array}$$

Therefore, $S = \frac{a}{r-1} \times \left\{ \begin{array}{l} n \text{ terms of } r+r^2+r^3+r^4, \text{\textcircled{C}c.} \\ -n \text{ terms of } 1+1+1+1, \text{\textcircled{C}c.} \end{array} \right.$

But $1+1+1+1, \text{\textcircled{C}c.} = n$

And $r+r^2+r^3+r^4, \text{\textcircled{C}c.} = \overline{r^n-1} \times \frac{r}{r-1}$.

Whence $S = \overline{r^n-1} \times \frac{r}{r-1} - n \times \frac{a}{r-1} = \text{sum required.}$

9. Required the sum (S) of n terms of the series

$\frac{1}{1} + \frac{3}{2} + \frac{7}{4} + \frac{15}{8} + \frac{31}{16}, \text{\&c.}$ the terms of which are

the successive sums of the geometrical progression

$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \text{\&c.}$

Let $a=1$ and $r=2$, then will $a + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \frac{a}{r^4},$

$\text{\textcircled{C}c.} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \text{\textcircled{C}c.}$

But the sums of 1, 2, 3, 4, $\text{\textcircled{C}c.}$ terms of this series are

$$\begin{array}{l} 1 \quad \frac{r-1 \times a}{r-1 \times 1} = r-1 \times \frac{a}{r-1} \\ 2 \quad \frac{r^2-1 \times a}{r-1 \times r} = r-\frac{1}{r} \times \frac{a}{r-1} \\ 3 \quad \frac{r^3-1 \times a}{r-1 \times r^2} = r-\frac{1}{r^2} \times \frac{a}{r-1} \\ 4 \quad \frac{r^4-1 \times a}{r-1 \times r^3} = r-\frac{1}{r^3} \times \frac{a}{r-1} \end{array}$$

$\mathcal{E}c.$ $\mathcal{E}c.$

Therefore $S = \frac{a}{r-1} \times \left\{ \begin{array}{l} n \text{ terms of } r+r+r+r, \mathcal{E}c. \\ -n \text{ terms of } \frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \mathcal{E}c. \end{array} \right.$

But $r+r+r+r, \mathcal{E}c. = nr,$

And $\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \mathcal{E}c. = \frac{r^n-1}{r-1 \times r^{n-1}},$

Whence $S = \frac{a}{r-1} \times nr - \frac{r^n-1}{r-1 \times r^{n-1}} = \text{sum required.}$

10. To find the sum (S) of the infinite series of the reciprocals of the triangular numbers, $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}, \&c.$

Let $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}, \mathcal{E}c. \text{ ad infinitum} = S.$

Or, $\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{2.5}, \mathcal{E}c. \dots \dots = S.$

Then $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \mathcal{E}c. \dots \dots = \frac{S}{2}.$

That is, $\frac{1}{1} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{5}, \mathcal{E}c. = \frac{S}{2}.$

$$\text{Or, } \left\{ \begin{array}{l} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \text{ } \mathcal{E}c. \\ - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \text{ } \mathcal{E}c. \end{array} \right\} = \frac{S}{2}$$

Whence, $\frac{S}{2} = \frac{1}{1}$; or $S = 2 = \text{sum required.}$

* 11. To find the sum of n terms of the series.

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15}, \text{ \&c.}$$

$$\text{Let } z = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \text{ } \mathcal{E}c. \text{ to } \frac{1}{n}.$$

$$\text{Then } z - \frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \text{ } \mathcal{E}c. \text{ to } \frac{1}{n}.$$

$$\text{And } z - \frac{1}{1} + \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \text{ } \mathcal{E}c. \text{ to } \frac{1}{n+1}.$$

* Let $\Sigma = \text{sum of an infinite number of terms, and } S = \text{sum of } n \text{ terms.}$

Then the formulæ for the sums of the reciprocals of figurate numbers may be exhibited as follows:

$\Sigma \quad S$

1	$\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}, \text{ } \mathcal{E}c. = \infty = n.$
2	$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \text{ } \mathcal{E}c. = \infty =$
3	$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}, \text{ } \mathcal{E}c. = \frac{2}{1} = \frac{2}{1} - \frac{2}{1} \times \frac{1}{n+1}.$
4	$\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20}, \text{ } \mathcal{E}c. = \frac{3}{2} = \frac{3}{2} - \frac{3}{2} \times \frac{1.2}{n-1.n-2}.$
5	$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35}, \text{ } \mathcal{E}c. = \frac{4}{3} = \frac{4}{3} - \frac{4}{3} \times \frac{1.2.3}{n+1.n+2.n+3}.$

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Therefore $\frac{1}{1} - \frac{1}{n+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \text{ \&c. to } \frac{1}{n} - \frac{1}{n+1}$

Or, $\frac{n}{n+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \text{ \&c. to } \frac{1}{n.n+1}$

Whence, $\frac{2n}{n+1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} \text{ \&c. to } \frac{2}{n.n+1}$

Or, $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} \text{ \&c. to } \frac{2}{n.n+1} \text{ (or } n \text{ terms)}$
 $= \frac{2n}{n+1} = \text{sum required.}$

12. Required the sum of the infinite series $\frac{1}{1.2.3.}$

$+ \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}, \text{ \&c.}$

Let $x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \text{ \&c. ad infinitum.}$

Then $x - \frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \text{ \&c. by transposition.}$

And $1 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \text{ \&c. by subtraction.}$

Or $1 - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6}, \text{ \&c. by transposition.}$

And $\frac{1}{2} = \frac{4}{1.4.3} + \frac{6}{2.9.4} + \frac{8}{3.16.5}, \text{ \&c. by subtraction.}$

Or $\frac{1}{2} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5} + \frac{2}{4.5.6}, \text{ \&c.}$

Whence $\frac{1}{2} \div 2 = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}, \text{ \&c.}$

Or, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}, \text{ \&c.} = \frac{1}{2} \div 2 =$

$\frac{1}{4} = \text{sum required.}$

13. To find the sum of n terms of the series $\frac{1}{1.2.3}$

$$+ \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}, \&c.$$

$$\text{Let } x = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +, \&c. \text{ to } \frac{1}{n.n+1}.$$

$$\text{Then } x - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \&c. \text{ to } \frac{1}{n.n+1}.$$

$$\text{And } x - \frac{1}{2} + \frac{1}{n+1.n+2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \&c. \text{ to } \frac{1}{n+1.n+2}.$$

$$\frac{1}{n+1.n+2}.$$

$$\text{Therefore } \frac{1}{2} - \frac{1}{n+1.n+2} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}, \&c.$$

(continued to n terms) by subtraction.

$$\text{Whence } \frac{1}{4} - \frac{1}{2.n+1.n+2} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}, \&c.$$

(continued to n term) by division.

$$\text{And consequently, } \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}, \&c. \text{ to } n$$

$$\text{terms} = \frac{1}{4} - \frac{1}{2.n+1.n+2} = \text{sum required.}$$

14. Required the sum (S) of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} -$

$\frac{1}{16}$, &c. continued *ad infinitum*.

$$\text{Let } x = \frac{1}{2} \text{ and } S = \frac{x}{1+x}.$$

$$\text{Then } \frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5, \&c.$$

$$\text{And } x = 1+x \times x - x^2 + x^3 - x^4, \&c.$$

$$= \frac{\{x - x^2 + x^3 - x^4 + x^5, \text{ \&c.}\}}{1+x}$$

$$\frac{x - x^2 + x^3 - x^4 + x^5, \text{ \&c.}}{+ x^2 - x^3 + x^4 - x^5, \text{ \&c.}}$$

$$= x + 0 + 0 + 0 + 0, \text{ \&c.}$$

Therefore $x = x$.

$$\text{And } x - x^2 + x^3 - x^4 + x^5, \text{ \&c.} = \frac{x}{1+x}.$$

Or $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}, \text{ \&c.} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3} = \text{sum required.}$

15. Required the sum of the infinite series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32}, \text{ \&c.}$

$$\text{Let } x = \frac{1}{2}, \text{ and } S = \frac{x}{1-x}.$$

$$\text{Then } \frac{x}{1-x} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5, \text{ \&c.}$$

$$\text{And } x = \frac{x}{1-x} \times x + 2x^2 + 3x^3 + 4x^4, \text{ \&c.}$$

$$= \frac{\{x + 2x^2 + 3x^3 + 4x^4, \text{ \&c.}\}}{1 - 2x + x^2}$$

$$\frac{x + 2x^2 + 3x^3 + 4x^4, \text{ \&c.}}{-2x^2 - 4x^3 - 6x^4, \text{ \&c.}}$$

$$+ x^3 + 2x^4, \text{ \&c.}$$

$$x + 0 + 0 + 0, \text{ \&c.}$$

Therefore $x = x$.

$$\text{And } x + 2x^2 + 3x^3 + 4x^4 + 5x^5, \text{ \&c.} = \frac{x}{1-x}$$

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Or $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32}, \&c. = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 2 = \text{sum required.}$

16. Required the sum (S) of the infinite series $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81}, \&c.$

$$\text{Let } x = \frac{1}{3}, \text{ and } \frac{z}{1-x} = S.$$

$$\text{Then } \frac{z}{1-x} = x + 4x^2 + 9x^3 + 16x^4 + 25x^5, \&c.$$

$$\text{And } z = (1-x)^3 \times x + 4x^2 + 9x^3 + 16x^4, \&c. \\ = x + x^2, \text{ as will be found by actual multiplication.}$$

$$\text{Therefore } x + x^2 = z.$$

$$\text{And consequently } x + 4x^2 + 9x^3 + 16x^4, \&c. = \frac{x \times 1 + x}{1-x}^3$$

$$\text{Or } \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81}, \&c. = \frac{\frac{1}{3} \times 1 + \frac{1}{3}}{1 - \frac{1}{3}}^3 = \frac{3}{2} = \text{sum required.}$$

17. Required the sum (S) of the infinite series $\frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2} + \frac{a+3d}{mr^3}, \&c.$

$$\text{Let } x = \frac{1}{r}, \text{ and } S = \frac{z}{m \cdot 1-x}.$$

$$\text{Then } \frac{z}{m \cdot 1-x} = \frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2} + \frac{a+3d}{mr^3}, \&c.$$

$$\text{Or } \frac{z}{1-x} = a + \frac{a+d}{r} + \frac{a+2d}{r^2} + \frac{a+3d}{r^3}, \&c.$$

$$\text{That is } \frac{z}{1-x} = a + a+d \times x + a+2d \times x^2 + a+3d \times x^3, \&c.$$

And $z = \frac{1}{1-x} \times a + a + dx + a + 2dx^2 + a + 3dx^3, \text{ \&c.}$
 $= \frac{1}{1-x} \times a + dx$, as will appear by actual multiplication.

Therefore $z = \frac{1}{1-x} \times a + dx$.

And consequently $\frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2}, \text{ \&c.} =$

$$\frac{\frac{1}{1-x} \times a + dx}{m \cdot \frac{1}{1-x^2}} = \text{sum required.}$$

EXAMPLES FOR PRACTICE.

1. To find the sum of n terms of the series $a + a - d + a - 2d + a - 3d + a - 4d, \text{ \&c.}$

$$\text{Ans. } \frac{n}{2} \times 2a - n - 1 \times d.$$

2. Required the sum of the infinite series $a + da + d^2a + d^3a + d^4a, \text{ \&c.}$ where d is a proper fraction.

$$\text{Ans. } \frac{a}{1-d}.$$

3. To find the sum of the infinite series $1 + 2^4x + 3^4x^2 + 4^4x^3 + 5^4x^4, \text{ \&c.}$

$$\text{Ans. } \frac{1 + 11x + 11x^2 + x^3}{(1-x)^6}.$$

4. Required the sum of the infinite series $1 + 3x + 6x^2 + 10x^3 + 15x^4, \text{ \&c.}$

$$\text{Ans. } \frac{1}{(1-x)^2}.$$

5. Required the sum of the infinite series $1 + 4x + 10x^2 + 20x^3 + 35x^4, \text{ \&c.}$

$$\text{Ans. } \frac{1}{(1-x)^3}.$$

6. Required the sum of the infinite series $\frac{1}{1.3} +$

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9}, \text{ \&c.}$$

$$\text{Ans. } \frac{1}{2}.$$

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7. Required the sum of 40 terms of the series
 $1 \times 2 + 3 \times 4 + 5 \times 6 + 6 \times 7$, &c. *Ans.* 22960.

8. Required the sum of the infinite series $\frac{1}{1.2.3.4}$
 $+ \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}$, &c. *Ans.* $\frac{1}{18}$.

9. Required the sum of n terms of the series
 $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}$, &c. *Ans.* $\frac{1}{18} - \frac{1}{3.n+1.n+2.n+3}$.

10. Required the sum of n terms of the series $\frac{1}{r}$
 $+ \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}$, &c. *Ans.* $\frac{1-p \times r}{r-1} - \frac{np}{r-1}$, where $p = \frac{1}{r}$.

11. Required the sum of the series $\frac{1}{1.4} + \frac{1}{2.5} +$
 $\frac{1}{3.6} + \frac{1}{4.7}$, &c. $\frac{1}{n \times 3 + n}$.
Ans. $\Sigma = \frac{11}{18}$, $S = \frac{n}{3+3n} + \frac{n}{12+6n} - \frac{n}{27+9n}$.

12. Required the sum of the series $\frac{1}{2.6} + \frac{1}{4.8} +$
 $\frac{1}{6.10}$, &c. $\frac{1}{2n.4+2n}$.
Ans. $\Sigma = \frac{3}{16}$, $S = \frac{5n+3n^2}{32+48n+16n^2}$.

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13. Required the sum of the series $\frac{1}{4 \cdot 8} + \frac{1}{6 \cdot 10} + \frac{1}{8 \cdot 12} + \dots$, &c. $\frac{1}{2+2n, 6+2n}$.

$$\text{Ans. } \Sigma = \frac{1}{48}, S = \frac{n}{16+16n} - \frac{n}{36+24n}.$$

14. Required the sum of the series $\frac{1}{3 \cdot 8} + \frac{1}{6 \cdot 12} + \frac{1}{9 \cdot 16} + \frac{1}{12 \cdot 20} + \dots$, &c. $\frac{1}{3n \cdot 4+4n}$.

$$\text{Ans. } \Sigma = \frac{1}{12}, S = \frac{n}{12+12n}.$$

* 15. Required the sum of the series $\frac{1}{2 \cdot 6 \cdot 4 \cdot 5} + \frac{1}{4 \cdot 8 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 10 \cdot 6 \cdot 7} + \dots$, &c. $\frac{1}{2n \cdot 4+2n \cdot 3+n \cdot 4+n}$.

$$\text{Ans. } \Sigma = \frac{7}{1152}, S = \frac{n}{24} \times \left(\frac{1}{-4 \cdot 1+n} + \frac{1}{8 \cdot 2+n} - \frac{5}{12 \cdot 3+n} + \frac{3}{16 \cdot 4+n} \right).$$

* A great variety of series, of different forms, may be found in other authors; but those which are here given will be sufficient for the learner's practice.

The names of the principal authors, who have written upon this subject, are as follow:

Archimedes; Arabes; D' Alembert; Barrow; Briggs; Nicholas, Daniel, John and James Bernoulli; Fermat; De Cartes; Clairaut; Condercat; Cotes; Dodson; Euler; Emerson; Fagnanus; Le Grange; Goldbach; Gregory; Halley; Harriot; Huddens; Huygens; Hutton; Kepler; Keil; Landen; Mac Laurin; De Lagney; Leibnitz; Lorgna; Lucas de Burgo; Manfredi; Monmort; De Moivre; Montono; Nichole; Newton; Oughtred; Riccati; Regnald; Saunderson; Sterling; Slufius; Simpson; Brook Taylor; Varignon; Vieta; Wallis; Waring; &c.

OF LOGARITHMS.

Logarithms are numbers so contrived and adapted to other numbers, that the sums and differences of the former shall correspond to, and shew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a series of numbers in arithmetical progression, answering to another series of numbers in geometrical progression.

Thus { 0. 1. 2. 3. 4. 5. Indices, or logarithms.

{ 1. 2. 4. 8. 16. 32. Geometric progression.

Or { 0. 1. 2. 3. 4. 5. Indices, or logarithms.

{ 1. 3. 9. 27. 81. 243. Geometric progression.

Or { 0. 1. 2. 3. 4. 5. Ind. or log.

{ 1. 10. 100. 1000. 10000. 100000. Geo. prog.

&c.

And, from hence, it is evident, that there may be as many kinds of indices, or logarithms, as there can be taken different kinds of geometric series; and that in any system, or table, of logarithms whatever, the logarithm of unity, or 1, will be nothing.

* The invention of logarithms is the undoubted right of Lord *Neper*, Baron of *Merchiston*, in *Scotland*, and is properly considered as one of the most useful and excellent discoveries of modern times. A table of these numbers was first published by him at *Edinburgh*, anno 1614, in a treatise entitled *Canon Mirificum Logarithmorum*; and as their great utility and extensive application, were sufficiently apparent, they were immediately received by all the learned throughout Europe. Mr. *Henry Briggs*, Savilian Professor of *Geometry* at *Oxford*,

It is, also, apparent, from the nature of these series, that, if any two indices be added together, their sum will be the index of that number which is equal to the product of the two terms, in the geometric progression, to which those indices belong.

Thus, the indices 2 and 3, being added together, are $=5$; and the numbers 4 and 8, or the terms corresponding with those indices, being multiplied together, are $=32$, which is the number answering to the index 5.

And, in like manner, if any one index be subtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which those indices belong.

Thus the index 6, minus the index 4, is $=2$; and the terms corresponding to those indices are 64 and 16, whose quotient is $=4$; which is the number answering to the index 2.

upon hearing of the discovery, set out on a visit to the noble inventor, and soon afterwards they jointly undertook the arduous task of computing new tables upon this subject, and reducing them to a more convenient form than that which was at first thought of. But Lord Neper dying before they were finished, the whole burden was laid upon Mr. Briggs, who with prodigious labour, and great skill, made an entire Canon, according to the new form, for all numbers from 1 to 20000, and from 9000 to 101000, to 14 places of figures, and published it at London in the year 1624, in a treatise entitled *Arithmetica Logarithmica*, with directions for supplying the intermediate chiliads.

This Canon was again published in Holland, by Adrian Vlacq, anno 1628, together with the logarithms of all the numbers, which Mr. Briggs had omitted; but he continued them only to 10 places of decimals. Mr. Briggs also computed the logarithms of the sines, tangents and secants, to every degree, and $\frac{1}{100}$ part of a degree of the whole quadrant; and subjoined

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus, the index, or logarithm of 4, in the above series, is 2; and if this number be multiplied by 3, the product will be $\equiv 6$; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus the index, or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be $\equiv 3$; which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice are such as are adopted to a geometric series increasing

them to the natural signs, tangents and secants, which he had before computed to 15 places of figures. And these tables, together with their construction and use, were first published in the year 1633, after Mr. Briggs's death, by Mr. Henry Gellibrand, under the title of *Trigonometria Britannica*.

Benjamin Ursinus has also given us a table of logarithms to every 10 seconds. And Mr. Wolf, in his *Mathematical Lexicon*, says, that one Van Lofer had computed them to every single second; but his untimely death prevented their publication.

A great number of other authors have treated of this subject, but as their numbers are frequently inaccurate, and incommodiouly disposed, they are now generally neglected. The tables in most repute at present, are those of Gardiner in 4to, first printed in the year 1742, and Sherwin in 8vo, first printed in the year 1705, where the logarithms of all numbers may be easily found from 1 to 1000000; and those of the signs, tangents, and secants, to any degree of accuracy required.

Dodson's *Antilogarithmic Canon* is likewise a very ingenious work, and is of great use for finding the numbers answering to any given logarithms.

in a tenfold proportion, as in the last of the above examples; and are those which are to be found, at present, in most of the common tables upon this subject.

And the distinguishing mark of this system of logarithms is, that the index, or logarithm, of 1 is 0; that of 10, 1; that of 100, 2; that of 1000, 3, &c. And, from hence it follows, that the logarithm of any number between 1 and 10 must be 0 and some fractional parts; and that of a number between 10 and 100 will be 1 and some fractional parts; and so on for any other number whatever.

And since the integral part of a logarithm is always thus readily found, it is usually called the *index*, or *characteristic*; and is commonly omitted in the tables; being left to be supplied by the operator himself, as occasion requires.

OF THE MAKING OF LOGARITHMS.

Whatever arithmetical progression we apply to a geometrical one, the terms of it are logarithms only to that series to which we apply them, and answer the end proposed only for those particular numbers; so that if we had logarithms adapted only to particular geometrical series, they would be but of little use. The great end and design of logarithms is the ease and expedition they afford in long calculations, by saving the laborious work of *multiplication*, *division*, and the *extraction of roots*; but this end would never be completely answered, unless logarithms could be adapted to the whole system of numbers, 1, 2, 3, 4, &c. And as here lay the excellence and merit of the contrivance, so also the difficulty. For the natural

system of numbers, 1, 2, 3, 4, &c. being an arithmetical, and not a geometrical series, seems rather fit to be made logarithms of, than to have logarithms applied to it. Yet this difficulty may be easily removed, by considering,

That though the whole system of natural numbers, 1, 2, 3, 4, &c. makes not one geometrical series, and cannot, by any means, be brought within such a series of determinate numbers, yet they may be brought so near to it, as to be within any assignable degree of approximation; which may be conceived, in general, thus: Suppose a fraction indefinitely small to be represented by x , and a geometrical series arising from 1, in the ratio of 1 to $1+x$, to be $1, 1+x, 1+x^2, 1+x^3, 1+x^4, \&c.$ Then must some of these terms come indefinitely near to coincide with all the natural numbers, 1, 2, 3, 4, &c.; because amongst numbers that arise by indefinitely small increments, some of them must exceed, or fall short, of any determinate number, by an indefinitely little excess or defect.

If, therefore, in the places of the terms of this series, that do approach indefinitely near to any of the natural numbers, we suppose these natural numbers themselves to be substituted, then will the series be a geometrical progression, to an exactness that may be called *indefinite*; because the approximation of its terms to the natural numbers, can never end, but goes on *in infinitum*.

And since this imagined geometric series comprehends, indefinitely near, the whole system of natural numbers, 1, 2, 3, 4, &c. so the indices of its terms comprehend a whole system of logarithms, which are adapted to this system of numbers, and may be extended to any length we please. For though the natural system of numbers make not, by themselves,

a complete geometrical series, yet they are conceived as a part of such a series, and their logarithms are the indices of their distances from unity in that series; or, more generally, they are the corresponding terms of an arithmetical series applied to that geometrical one.

But, again, it must be observed, that an indefinitely small fraction cannot be assigned; and, therefore, in the actual construction of logarithms, we must be contented with a determinate degree of approximation. Whence, accordingly as we take x , so in the series $1, 1+x, 1+x^2, 1+x^3, 1+x^4$, &c. the approximation of its terms to the natural numbers will be in different degrees of exactness: for the less x is, the nearer will be the approximation; but then the more are the number of involutions of $1+x$, necessary to come within any determinate degree of nearness to the natural number assigned.

Thus then we may conceive the possibility of making logarithms to all the natural numbers, 1, 2, 3, 4, &c. to any determinate degree of exactness; viz. by assigning a very small fraction for x , and actually raising a series, in the ratio of 1 to $1+x$, and taking for the natural numbers such terms of that series as are the nearest to them, and their indices for the logarithms. But then, to construct logarithms in this manner, to such an extent of numbers, and degree of exactness, as would be necessary to make them of any considerable use, is next to impossible, because of the almost infinite labour and time it would require. This, however, is an introduction for understanding the method of the *noble inventor*, who undoubtedly first took the hint of making logarithms from the consideration of the indices of a geometrical series; and by means

of the principles and known properties of these progressions, he first formed his tables, and adapted them to the practical purposes intended.

P R O B L E M. I.

To find the logarithm of any of the natural numbers, 1, 2, 3, 4, &c. according to the method of NEPER.

R U L E.

1. Take the geometrical series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical series 1, 2, 3, 4, 5, &c. as logarithms.
2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series betwixt which the number proposed lies.
3. Between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.
4. Find as many arithmetical means, in the same order as you found the geometrical ones, and the last of these will be the logarithm answering to the number required.

E X A M P L E:

Let it be required to find the logarithm of 9.

Here the numbers between which 9 lies are 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0;

therefore $\frac{1+0}{2} = .5$ is the arithmetical mean. And

$\sqrt{1 \times 10} = \sqrt{10} = 3.1622777 =$ geometric mean;
whence the logarithm of 3.1622777 is .5.

Secondly, the log. of 10 is 1, and the log. of 3.1622777 is .5; therefore $\frac{1+.5}{2} = .75 =$ arithmetical mean. And $\sqrt{10 \times 3.1622777} = 5.6234132 =$ geometric mean; whence the log. of 5.6234132 is .75.

Thirdly, the log. of 10 is 1, and the log. of 5.6234132 is .75; therefore $\frac{1+.75}{2} = .875 =$ arithmetical mean. And $\sqrt{10 \times 5.6234132} = 7.4989421 =$ geometric mean; whence the log. of 7.4989421 is .875.

Fourthly, the log. of 10 is 1, and the log. of 7.4989421 is .875; therefore $\frac{1+.875}{2} = .9375 =$ arithmetical mean. And $\sqrt{10 \times 7.4989421} = 8.6596431 =$ geometric mean; whence the log. of 8.6596431 is .9375.

Fifthly, the log. of 10 is 1, and the log. of 8.6596431 is .9375; therefore $\frac{1+.9375}{2} = .96875 =$ arithmetical mean. And $\sqrt{10 \times 8.6596431} = 9.3057204 =$ geometric mean; whence the log. of 9.3057204 is .96875.

Sixthly, the log. of 8.6596431 is .9375, and the log. of 9.3057204 is .96875; therefore $\frac{.9375 + .96875}{2} =$

.953125 = arithmetical mean. And

$\sqrt{8.6596431 \times 9.3057204} = 8.9768713 =$ geometric mean; whence the log. of 8.9768713 is .953125.

And, proceeding in this manner, after 25 extractions, the logarithm of 8.9999998 will be found to be .9542425; which may be taken for the logarithm of 9, because it differs from it only by $\frac{1}{1000000}$, and is therefore sufficiently exact for all practical purposes.

And in the same manner the logarithms of almost all the prime numbers were found; a work so incredibly laborious, that the unremitting industry of several years was scarcely sufficient for its accomplishment.

PROBLEM II.

To determine the hyperbolic logarithm (L) of any given number (N).

The hyperbolic logarithm of any number, is the index of that term of the logarithmic progression, agreeing with the proposed number, multiplied by the excess of the common ratio above unity.

Let, therefore, $(1+x)^n$ be that term of the logarithmic progression, $1, 1+x, 1+x^2, 1+x^3, 1+x^4, \&c.$ which is equal to the required number (N).

Then will $(1+x)^n = N$, and $1+x = N^{\frac{1}{n}}$; and if $1+y$ be put $= N$, and $m = \frac{1}{n}$, we shall have $1+x$

$$= N^{\frac{1}{n}} = (1+y)^{\frac{1}{n}} = 1 + my + m \times \frac{m-1}{2} y^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3, \&c.$$

And, consequently, $x = my + m \times \frac{m-1}{2} y^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3, \&c.$ where m being rejected in the

factors $m-1, m-2, m-3, \&c.$ as being indefinitely small in comparison of $1, 2, 3, \&c.$ the equation will become $x = my - \frac{my^2}{2} + \frac{my^3}{3} - \frac{my^4}{4}, \&c.$

Hence $\frac{x}{m} (nx=L) = x - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}$,
 &c. = hyperbolic logarithm of N , as was required.

P R O B L E M III.

The hyperbolic logarithm (L) of a number being given, to find the number (N) itself, answering thereto.

Let $\overline{1+x}^n$ be that term of the logarithmic progression, $1, \overline{1+x}^1, \overline{1+x}^2, \overline{1+x}^3, \overline{1+x}^4$, &c. which is equal to the required number N .

Then, because $\overline{1+x}^n$ is universally $= 1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. we shall also have
 $1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. = N .

But because n , from the nature of logarithms, is here supposed indefinitely great, it is evident that the numbers connected to it by the sign —, may all be rejected, as far as any assigned number of terms, being indefinitely small in comparison of n .

Therefore, by throwing out 1, 2, 3, &c. from the factors $\frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}$, &c. we shall have $1 + nx + \frac{n^2 x^2}{2} + \frac{n^3 x^3}{2.3} + \frac{n^4 x^4}{2.3.4}$, &c. = N .

But $nx (=L)$ is the hyperbolic logarithm of $\overline{1+x}^1$, or N , by what has been before specified; and therefore $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4}$, &c. = N = number required.

P R O B L E M IV.

To determine the hyperbolic logarithm (L) of any given number (N), by an universally converging series.

The series $y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}$, &c. is the most easy and natural that can be obtained; but, in determining the logarithms of large numbers, it is but of little use, since, in all such cases, it diverges instead of converging.

Let, therefore, the number whose logarithm you would find, be denoted by $\frac{1}{1-y}$, and also let $(1+x)^n$ be the term of the logarithmic progression agreeing with the proposed number.

$$\text{Then } (1+x)^n = \frac{1}{1-y}; \text{ or } 1+x = \frac{1}{(1-y)^{\frac{1}{n}}} =$$

$$(1-y)^{-\frac{1}{n}} = (1-y)^m \left(\text{by putting } m = -\frac{1}{n} \right) = 1 - my$$

$$+ m \times \frac{m-1}{2} y^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3, \text{ \&c.}$$

And, if m be rejected in the factors $m-1$, $m-2$, $m-3$, as before, our equation will become $1+x = 1 - my - \frac{my^2}{2} - \frac{my^3}{3} - \frac{my^4}{4}$ &c.

$$\text{Whence } y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} \text{ \&c.} = -\frac{x}{m} = nx =$$

hyperbolic logarithm of $\frac{1}{1-y}$; which series, it is manifest, will constantly converge, let the value of $\frac{1}{1-y}$ be ever so great; because y will always be less than unity.

But it is to be observed, that this series, except in its signs, has exactly the same form with that

above found for the logarithm of $1+y$, and that, if both of them be added together, the series $2y + \frac{2y^3}{3} + \frac{2y^5}{5} + \frac{2y^7}{7}$, &c. thence arising, will be more simple than either of them, since one half of the terms will be intirely destroyed thereby.

Since, therefore, the sum of the logarithms of any two numbers is equal to the logarithm of the product of those numbers, it is manifest that $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, &c. will truly express the logarithm of

$\frac{1+x}{1-x}$; which series converges still faster than $x +$

$\frac{x^2}{2} + \frac{x^3}{3}$, &c. not only because the even powers are here destroyed, but because x , in finding the logarithm of any given number (N) will have a less value.

And, in order to determine what this value of x must be, make $\frac{1+x}{1-x} = N$, and then x will be found

$= \frac{N-1}{N+1}$; but if the quantity proposed $\left(\frac{P}{Q}\right)$ be a

fraction, instead of a whole number, make $\frac{1+x}{1-x} =$

$\frac{P}{Q}$, and you will have $x = \frac{P-Q}{P+Q}$; and either of

these values being substituted in the foregoing series $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$, &c. will give the hyperbolic logarithm of the number required.

Now, by finding *Neper's* logarithm of any number, according to the foregoing method, *Briggs's*,

or the common logarithm of the same number, may be found as follows :

Briggs's logarithm of any number, is to *Neper's* logarithm of the same number, as *Briggs's* logarithm of 10, is to *Neper's* logarithm of 10.

But *Briggs's* logarithm of 10 is 1, and *Neper's* logarithm of 10 is 2.302585093; and, therefore, if *Briggs's*, or the common logarithm of any number, be denoted by C.L., and *Neper's*, or the hyperbolic logarithm of the same number, by H.L. we shall have $2.302585093 : 1 :: \text{H.L.} : \text{C.L.}$; or

$$\text{H.L.} \times \frac{1}{2.302585093} = \text{H.L.} \times .4342944819 = \text{C.L.},$$

as was required*.

* There are, besides these, many other ingenious methods, which later writers have discovered, for finding the logarithms of numbers, in a much easier way than their original inventor; but as they cannot be understood without a knowledge of some of the higher branches of the mathematics, I have thought proper to omit them, and must beg leave to refer the reader to those works that are written expressly upon the subject.

It would, likewise, much exceed the limits of this compendium, to point out all the peculiar artifices that are made use of for constructing an entire table of these numbers; such as those of *Gardiner*, *Sherwin*, and others, who have treated on this subject.

It will be sufficient to observe, that the logarithms of all the prime numbers being had, those of the composite numbers may be found only by means of addition and subtraction. Thus, $L.4 = 2L.2$; $L.10 = L.2 + L.5$; $L.5 = L.10 - L.2$; $L.6 = L.2 + L.3$, and so on for any other of these numbers.

OF THE METHOD OF USING A TABLE OF LOGARITHMS.

Having explained the method of making a table of the logarithms of numbers greater than unity, the next thing to be done is, to shew how the logarithms of fractional quantities may be found. And, in order to this, it may be observed, that as we have hitherto supposed a geometric series to increase from an unit on the right hand, so we may now suppose it to decrease from an unit towards the left; and the indices, in this case, being made negative, will still exhibit the logarithms of the terms to which they belong.

Thus, Log. $-3 \ -2 \ -1 \ 0 \ +1 \ +2 \ +3$, &c.
 Num. $\frac{1}{1000} \ \frac{1}{100} \ \frac{1}{10} \ 1 \ 10 \ 100 \ 1000$, &c.

Where $+1$ is the logarithm of 10, and -1 the logarithm of $\frac{1}{10}$; $+2$ the logarithm of 100, and -2 the logarithm of $\frac{1}{100}$, &c.

And from hence it appears, that all numbers, consisting of the same figures, whether they be integral, fractional, or mixed, will have the decimal parts of their logarithms the same.

Thus, the logarithm of 5874 being 3.7689339, the logarithms of $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. part of it will be as follows:

Num.	Logarithms.
5 8 7 4	3.7 6 8 9 3 3 9
5 8 7.4	2.7 6 8 9 3 3 9
5 8.7 4	1.7 6 8 9 3 3 9
5.8 7 4	0.7 6 8 9 3 3 9
.5 8 7 4	-1.7 6 8 9 3 3 9
.0 5 8 7 4	-2.7 6 8 9 3 3 9
.0 0 5 8 7 4	-3.7 6 8 9 3 3 9

From this it also appears, that the *index*, or *characteristic*, of any logarithm, is always one less than the number of figures which the natural number consists of; and this index is constantly to be placed on the left hand of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a line drawn before it, like a negative quantity in algebra.

Thus, a number having 1, 2, 3, 4, 5, &c. integer places.

The index of its log. is 0, 1, 2, 3, 4, &c.

And a fraction having a digit in the place of primes, seconds, thirds, fourths, &c.

The index of its logarithm will be -1 , -2 , -3 , -4 , &c.

It may also be observed, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative; and all operations are performed by them, in the same manner as by negative and affirmative quantities in algebra.

In taking out of a table the logarithm of any number, not exceeding 10000, we have the decimal part by inspection; and if to this the proper characteristic be affixed, it will give the complete logarithm required.

But if the number, whose logarithm is required, be above 10000, then find the logarithm of the two nearest numbers to it, that can be found in the table, and say, as their difference : the difference of their logarithms :: the difference between the nearest number and that whose logarithm is required : the dif-

ference of their logarithms, *nearly*; and this difference being added to, or subtracted from, the nearest logarithm, according as it is greater or less than the required one, will give the logarithm required, *nearly*.*

Thus, let it be required to find the logarithm of 367182.

The decimal part of 3671 is, by the table 5647844; and of 3672 is 5649027;

$$\therefore \text{The } \begin{cases} 367100 & \text{is } 5.5647844 \\ \text{log. of } 367200 & \text{is } 5.5649027 \end{cases}$$

Their diff. 100 .0001183 diff.

$$\begin{array}{l} \text{Nearest } N^{\circ} \{ 367200 \\ \text{Given } N^{\circ} \{ 367182 \end{array}$$

18 diff.

Therefore $100 : .0001183 :: 18 : .0000212$,

And $5.5649027 - .0000212 = 5.5648815 = \text{logarithm of } 367182 \text{ nearly.}$

If the number consists both of integers and fractions, or is entirely fractional, find the decimal part of the logarithm as if all its figures were integral; and this, being prefixed to the proper characteristic, will give the logarithm required.

* This method being founded on the supposition, that the logarithms of all numbers between 367100 and 367200, increase, or decrease, equally, according to their distance from 367100 or 367200, is not strictly true, but nearly so; and the greater any numbers are with respect to their difference, the nearer will those differences be proportional. And, therefore, though this will not give the exact logarithm, yet it will be a very near approximation, and is sufficiently exact for most practical purposes.

To find the logarithm of a proper fraction ; subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought ; which, being that of a decimal fraction, must always have a negative index.

And to find the logarithm of a mixed number, reduce the given number into an improper fraction ; then subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought.

In finding the number answering to any given logarithm, the index, if *affirmative*, will always shew how many integral places the required number consists of ; and, if *negative*, in what place of decimals the first, or significant figure, stands ; so that if the logarithm can be found in the table, the number answering to it will always be had by inspection.

But, if the logarithm cannot be exactly found in the table, find the next greater, and the next less, and then say, As the difference of these two logarithms : the difference of the numbers answering to them :: the difference between the given logarithm and the nearest tabular logarithm : a fourth number ; which added to, or subtracted from, the natural number answering to the nearest tabular logarithm, according as that logarithm is less or greater than the given one, will give the number required, *nearly*.

Thus, let it be required to find the natural number answering to the logarithm 5.5648815.

The next less and greater logarithms, in the table, are

5.5647844	} The numbers	{ 367100
5.5649027		

Their diff. .0001183. *100 diff.*

And $5.5649027 - 5.5648815 = .0000212$,
 Therefore $.0001183 : 100 :: .0000212 : 18$ nearly.
 Whence $367200 - 18 = 367182 =$ number required*.

MULTIPLICATION by LOGARITHMS.

R U L E.

Add the logarithms of the factors together, and their sum will be the logarithm of the product required.

Observing to add what is to be carried from the decimal part of the logarithm to the sum of the affirmative indices :

And that the difference between the affirmative and negative indices, is to be taken for the index to the logarithm of the product.

E X A M P L E S :

1. Let the number 256 be multiplied by 4.

$$\text{The log. of } 256 = 2.4082400$$

$$\text{The log. of } 4 = 0.6020600$$

$$\text{The product} = 1024 \dots 3.0103000$$

* Directions, at large, for the using of logarithms, may be found in most of the common tables upon this subject.—
Sherwin's Mathematical tables, of the Edition 1741, or 1742, are reckoned the most correct and convenient, for practical purposes, of any now extant.

2. Let the number 8.5 be multiplied by 10.

$$\text{The log. of } 8.5 = 0.9294189$$

$$\text{The log. of } 10 = 1.0000000$$

$$\text{The product} = 85 \dots 1.9294189$$

3. Let the number 46.75 be multiplied by .3275.

$$\text{The log. of } 46.75 = 1.6697816$$

$$\text{The log. of } .3275 = -1.5152113$$

$$\text{The product} = 15.31 \dots 1.1849925$$

4. Multiply 3.768, 2.053, and .007693 continually together.

$$\text{The log. of } 3.768 = 0.5761109$$

$$\text{The log. of } 2.053 = 0.3123889$$

$$\text{The log. of } .007693 = -3.8860957$$

$$\text{The product} = .059511 \dots -2.7745955$$

5. Multiply .5, .4, and .12, continually together,

$$\text{The log. of } .5 = -1.6989700$$

$$\text{The log. of } .4 = -1.6020600$$

$$\text{The log. of } .12 = -1.0791812$$

$$\text{The product} = .024 \dots -2.3802112$$

DIVISION *by* LOGARITHMS.

R U L E.

From the logarithm of the dividend subtract the logarithm of the divisor, and the number agreeing to the remainder will be the quotient required.

But observe to change the index of the divisor from negative to affirmative, or from affirmative to negative, and then the difference of the affirmative indices must be taken for the index to the logarithm of the quotient.

And, also, when an unit is borrowed, in the left hand place of the decimal part of the logarithm, add it to the index of the divisor; but if it be negative subtract it; and let the index arising from thence be changed and worked with as before.

E X A M P L E S:

1. Let the number 56 be divided by the number 4.

$$\text{The log. of } 56 = 1.7481880$$

$$\text{The log. of } 4 = 0.6020600$$

$$\text{The quotient} = 14. \dots 1.1461280$$

2. Let the number 50.75 be divided by the number .25.

$$\text{The log. of } 50.75 = 1.7054360$$

$$\text{The log. of } .25 = -1.3979400$$

$$\text{The quotient} = 203. \dots 2.3074960$$

3. Let the number 24 be divided by the number 80.

$$\begin{array}{l} \text{The log. of } .24 = -1.3802112 \\ \text{The log. of } .80 = \quad 1.9030900 \end{array}$$

$$\text{The quotient } .003 \dots -3.4771212$$

4. Let the number .01265 be divided by the number .35.

$$\begin{array}{l} \text{The log. of } .01265 = -2.1020905 \\ \text{The log. of } .35 = -1.7403627 \end{array}$$

$$\text{The quotient } = .023 \dots -2.3617278$$

INVOLUTION *by* LOGARITHMS*.

R U L E.

1. Seek the logarithm of the given number in the table.
2. Multiply the logarithm, thus found, by the index of the proposed power.
3. Find the number corresponding to the product, and it will be the power required.

Note, In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative.

But what is to be carried from the decimal part of the logarithm will always be affirmative :

* The rule of proportion is performed by adding the logarithms of the two last terms, and subtracting the logarithm of the first.

And therefore their difference will be the index of the product; and is constantly to be made of the same kind with the greater.

E X A M P L E S :

1. Required the second power of the number 3.874.

$$\text{The log. of } 3.874 = 0.5881596$$

$$\text{The index} = \underline{\quad 2 \quad}$$

$$\text{The power} = 15.01 \dots 1.1763192$$

2. Required the third power of the number 2.768.

$$\text{The log. of } 2.768 = 0.4421661$$

$$\text{The index} = \underline{\quad 3 \quad}$$

$$\text{The power} = 21.21 \dots 1.3264983$$

3. Required the third power of the number .7916.

$$\text{The log. of } .7916 = -1.8985058$$

$$\text{The index} = \underline{\quad 3 \quad}$$

$$\text{The power} = .4961 \dots -1.6955174$$

4. Required the twelfth power of the number 1.539.

$$\text{The log. of } 1.539 = 0.1872386$$

$$\text{The index} = \underline{\quad 12 \quad}$$

$$\text{The power} = 176.6 \dots 2.2468632$$

EVOLUTION *by* LOGARITHMS.

R U L E.

1. Seek the logarithm of the given number, in the table.

2. Divide the logarithm, thus found, by the denominator of the index of the root proposed.

3. Find the number corresponding to this quotient, and it will be the root required.

Note, When the index of the logarithm, to be divided, is negative, and does not exactly contain the divisor; increase it by such a number as will make it exactly divisible, and carry the units borrowed, as so many tens, to the left hand place of the decimal, and then divide as in whole numbers.

E X A M P L E S :

1. Required the square root of the number 225.

$$\text{The log. of } 225 = 2.3521825$$

$$\text{Therefore } 2) 2.3521825$$

$$\text{The root} = 15 \dots 1.1760912$$

2. Required the square root of the number 1501.

$$\text{The log. of } 1501 = 3.1763807$$

$$\text{Therefore } 2) 3.1763807$$

$$\text{The root} = 38.74 \dots 1.5881903$$

MISCELLANEOUS QUESTIONS.

3. What is the cube root of the number .166375?

*The log. of .166375 = -1.2210881**Therefore 3) -1.2210881**The root = .55 ... -1.7403627*

4. What is the square root of the number .08162?

*The log. of .08162 = -2.9117966**Therefore 2) -2.9117966**The root = .2857 ... -1.4558983*

5. What is the twelfth root of the number 176.6?

*The log. of 176.6 = 2.2469907**Therefore 12) 2.2469907**The root = 1.5391872492*

MISCELLANEOUS QUESTIONS.

1. A person being asked what o'clock it was, answered, that it was between 8 and 9, and that the hour and minute hands were exactly together; what was the time?

*b.**Ans. 8 : 43 : 38 $\frac{2}{11}$.*

2. Divide the number 50 into two such parts, that
- $\frac{3}{4}$
- of one part, added to
- $\frac{5}{6}$
- of the other, may make 40.

Ans. 20 and 30.

3. What two numbers are those, whose difference is 12, and their squares equal to each other?

Ans. +6 and -6.

4. There is a certain number, consisting of two places, which is equal to the difference of the squares

of its digits; and if 36 be added to it the digits will be inverted; quære the number? *Ans.* 48.

5. Given $x^3 + y^3 = 31$, and $y^3 + x^2 = 17$; to find x and y . *Ans.* $x = 3$ and $y = 2$.

6. Given $y^3 - xy = 666$, and $x^3 + xy = 406$; to find x and y . *Ans.* $x = 7$ and $y = 9$.

7. Given the sum of three numbers, in harmonical proportion, $= 26$, and their continued product $= 576$; to find the numbers. *Ans.* 12, 8, and 6.

8. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5 respectively? *Ans.* 2 and 10.

9. To find that number whose cube being subtracted from its square shall leave the greatest remainder possible? *Ans.* $\frac{2}{3}$.

10. Required to find the least 3 whole numbers, so that $\frac{1}{2}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{6}$ of the third, shall be all equal to each other.

Ans. 280, 294, and 300.

11. Given $xx^3 + xx^3 = 290$, and $x^4 + x^4 = 641$; to find x and x . *Ans.* $x = 5$, and $x = 2$.

12. Given the sum of three numbers in continued geometrical progression $= 39$, and the sum of their squares $= 819$; to find the numbers.

Ans. 3, 9, 27.

13. Required the fewest number of weights, and the weight of each, that shall weigh from one pound to 29 hundred weight.

Ans. 1, 3, 9, 27, 81, 243, 729, and 2155.

14. Required two numbers such, that their sum shall be equal both to their product and the difference of their squares.

Ans. 2.618034 and 1.618034.

15. It is required to find 4 affirmative integers such, that the square of the greatest may be equal to the sum of the squares of the other three.

Ans. 3, 4, 12, and 13.

196 MISCELLANEOUS QUESTIONS.

16. If money be lent, at three per cent.

To those who chuse to borrow,

In what time shall I be worth a pound,

If I lend a crown to-morrow?

Ans. 46.90036 years, allowing comp. int.

17. Required the two least nonquadrate numbers, x and y , such, that $x^2 + y^2$, and $x^3 + y^3$ shall be both square numbers.

Ans. $x=364$, and $y=273$.

18. There are three numbers in geometrical proportion such, that if the mean be subtracted from the sum of the two extremes, the remainder multiplied by the sum of the said two extremes will be 91; but if that remainder be multiplied by the sum of all the three numbers, the product will be 133; it is required to find the three numbers by a simple equation.

Ans. 4, 6, and 9.

19. To determine two numbers whose sum shall be a cube, but their product and quotients squares.

Ans. 4 and 4, 100 and 25, 900 and 100.

20. Required that arithmetical progression whose number of terms is 10, sum of the terms 185, and the sum of the cubes of the terms 104525.

Ans. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

21. To divide a given number (N) into 4 such parts, that if any other number (n) be added to the first part, deducted from the second, multiplied by the third, and the fourth part divided thereby, the sum, difference, product, and quotient, shall be all equal to each other.

Ans. $\frac{Nn}{n+1} - n$, $\frac{Nn}{n+1} + n$, $\frac{N}{n+1}$, and $\frac{Nn \times n}{n+1} =$

[parts required.]

22. Given $x^3y + y^3x = 512500$, and $x^2y - y^2x = 2500$; to find x and y .

Ans. $x=25$ and $y=20$.

23. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x , y , and z .

Ans. $x=3$, $y=1$, and $z=2$.

24. To find two numbers in the ratio of 5 to 7, and which being respectively divided by 9 and 13, shall leave 3 and 8 for remainders.

Ans. 210 and 294.

25. To find three numbers such, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be $=62$; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third $=47$; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third $=38$.

Ans. 24, 60, and 120.

26. Given $x + \frac{y}{2} = 357$, $y + \frac{z}{3} = 476$, $z + \frac{w}{4} = 595$, and $w + \frac{x}{5} = 714$; to find x , y , z , and w .

Ans. $x=192$, $y=334$, $z=426$, and $w=676$.

27. To find four numbers x , y , z , and w , having the product of every three given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Ans. $x=3$, $y=7$, $z=11$, and $w=20$.

28. To find four numbers in geometric proportion, whose sum is 15, and the sum of their squares 85.

Ans. 1, 2, 4, 8.

29. To find three numbers, x , y , and z , when the product of each by the sum of the other two are given; viz. $x \times y + z = 48$, $y \times x + z = 39$, and $z \times x + y = 63$.

Ans. $x=4$, $y=3$, and $z=9$.

30. What number is that, which, being any how divided, the square of one part, when added to the other part, shall always be a square number?

Ans. 1 only.

31. Given $y^3 + z = 127$, $y^3 + x = 135$, and $x^3 + y^3 + z^3 = 1133$; to find x , y , and z .

Ans. $x=10$, $y=5$, and $z=2$.

32. Given $x^2 + xy = 108$, $y^2 + yz = 69$, and $z^2 + xz = 580$; to find x , y , and z .

Ans. $x=9$, $y=3$, and $z=20$.

198 MISCELLANEOUS QUESTIONS.

33. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x , y , z .

Ans. $x=10$, $y=17$, and $z=22$.

34. To find the least number, which being divided by 6, 5, 4, 3, and 2, shall leave the remainders 5, 4, 3, 2, and 1 respectively. *Ans.* 59.

35. To find three numbers such, that the sum or difference of any two of them shall be square numbers. *Ans.* 1873432, 2399057, and 2288168.

36. To find two square numbers such, that their sum may be a square, and their difference a cube, and the side of the said square and cube equal to each other.

Ans. $\frac{783}{15625}$ and $\frac{414}{15625}$.

37. To determine the number of fifteens that can be made out of a common pack of 52 cards.

Ans. 17264.

38. To find a fraction such, that being taken from its reciprocal the remainder shall be a square.

Ans. Find such a fraction as that its biquadrate being added to 4 is a square, and it will answer the question.

39. Given $x^2+xy+y^2=1087$, and $x^4+x^3y^3+y^4=4577295$; to find x and y .

Ans. $x=21$ and $y=17$.

40. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$; to find x , y , and z .

Ans. $x=41$, $y=28$, and $z=9$.

41. Given $x+y=152$, and $\sqrt{x-y}^2 \times \sqrt{x-y}^2 = 8192$; to find x and y . *Ans.* $x=108$, and $y=44$.

42. To find three numbers such, that if to the square of each the product of the other 2 be added, the sums shall be squares. *Ans.* 73, 9, 328.

43. Let the number of cards in a pack (p) be distributed into any number of heaps (n), by laying as many cards upon the bottom heap as are suffi-

MISCELLANEOUS QUESTIONS. 199

cient to make up its number q ; then by having the number of cards remaining in the dealer's hand, (r) and the number of heaps (n) given, it is required to find the sum of all the bottom cards.

Ans. $q + 1 \times n + r - p = \text{sum required.}$

44. To find 3 numbers such, that if each be subtracted from the cube of their sum, the remainders shall be cubes.

Ans. $\frac{13851}{85184}, \frac{19467}{85184}, \text{ and } \frac{18954}{85184}.$

45. Given $x^x = 123456789$ to find x .

Ans. $x = 8.6400268.$

46. Given the cycle of the sun 18, the golden number 8, and the Roman indiction 10; to find the year.

Ans. 1717.

47. To find 3 cube numbers such, that their sum shall be both a square and a cube number; and if that sum be squared it shall be a cube, and if it be cubed it shall be a square.

Ans. $\frac{x^6}{8}, \frac{8x^6}{27}, \frac{125x^6}{216};$ where x may be any num-

ber whatever; if it be $= 216$ they will be whole numbers.

48. To find 3 numbers such, that if each be added to the cube of their sum, the sums shall be cubes.

Ans. $\frac{23625}{157464}, \frac{1538}{157464}, \frac{18577}{157464}.$

49. With guineas and moidores, the fewest, which way,

Three hundred and fifty-one pounds can you pay.

If paid every way 'twill admit of, what sum

Do the pieces amount to?—my fortune's to come.

Ans. 9 guineas and 253 moidores; and 37 ways, which is $= 12987$ l.

200 MISCELLANEOUS QUESTIONS.

50. Given $y^{\frac{1}{2}} = z^{\frac{2}{3}} = 100$; to find x and z .

Ans. $x=47.706$ and $z=1.42$.

51. Given $44000x^2 + 1 = z^2$; to find x and y in whole numbers.

Ans. $x=40482981221781$ and $z=8491781781142001$.

52. To find three whole numbers such, that the excess of the greatest above the middle number, shall be to the excess of the middle number above the least, as 3 to 1; and also that the sum of every two of these shall be a square.

Ans. 1362, 402, 82; or $4^n \times 1362$, $4^n \times 402$, and $4^n \times 82$; where n is any affirmative integer.

53. Given $x+y=a(2)$, and $x^2+y^2=b(32)$, to find x and y by quadratics.

Ans. $x=1.4697175$ and $y=.5302824$.

54. Given $x^y=500$, and $y^x=300$; to find x and y .

Ans. $x=4.6914$ and $y=5.5102$.

55. Given $xy \times x+z^2=300$, $xz \times y+z^2=1296$, and $yz \times x+y^2=432$; to find x , y , and z .

Ans. $x=1$, $y=3$, $z=9$.

56. Given $w^3+x+y+z=57$, $w+x^3+y+z=2763$, $w+x+y^3+z=1353$, and $w+x+y+z^3=153$; to find x , y , z , and w .

Ans. $x=14$, $y=11$, $z=5$, and $w=3$.

57. Given $x+y=1750$, $xz+yv=22708$, $xv+yz=12292$, and $xzv+vzy=159252$; to find x , y , z , and v .

Ans. $x=1743$, $y=7$, $z=13$, and $v=7$.

58. Given $5x+7y+9z=93256$; to find all the different solutions in affirmative integers which the equation will admit of.

Ans. 13801148.

59. To find a square number such, that the sum of all its aliquot parts shall be a square number.

Ans. 2401.

MISCELLANEOUS QUESTIONS. 201

60. To find two square numbers such, that either of them, when added to its aliquot parts, shall make the same sum. *Ans.* 106276 and 165649.

61. To find three cube numbers such, that their sum may be both a square and cube number.

Ans. 1, $\frac{2048383}{274625}$, and $\frac{15252992}{274626}$.

62. To find 4 whole numbers such, that the difference of every two shall be a square number.

Ans. 1873432, 2288168, 2399057, and 6560657.

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